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Maximin, Viability and Sustainability

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Abstract

The maximin criterion defines the highest utility level which can be sustained in an intergenerational equity perspective. The viability approach characterizes all the economic trajectories sustaining a given, not necessarily maximal, utility level. In this paper, we exhibit the strong links between *maximin* and *viability*: We show that the value function of the maximin problem can be obtained in the viability framework via a static optimization problem under constraints. This result allows us to extend the maximin approach beyond optimality and characterize the sustainability of other economic trajectories. In particular, we show how the maximin value and viability kernel can be combined as sustainability indicators along any economic trajectory. Attention is especially paid to positive net investment at maximin prices, which is shown to be necessary to maintain the productive capacities of the economy. The Dasgupta-Heal-Solow model illustrates the assertions.

Key words: Sustainability, Maximin, Viability, Dynamics, Optimality.

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1 Introduction

Discounted utility, the main criterion used in economics for intertemporal choices, defines the Net Present Value of the economy (Weitzman, 2003), and provides theoretical basis to compute the National Net Product index used for national accounting (Weitzman, 1976; Dasgupta and Mäler, 2000). The sustainability issue is also addressed using criteria (Heal, 1998). A challenge to operationalize sustainability is to determine index to measure it. Such index should emerge from the theoretical frameworks provided by sustainability criteria, and would be the basis of sustainability accounting (Cairns, 2008; Dasgupta, 2009). A first attempt to tackle this challenge is to complete the National Net Product by accounting for natural resources depreciation and non-market goods, to obtain a "comprehensive" accounting (Repetto et al., 1989; Asheim, 1994; Weitzman and Löfgren, 1997; Cairns, 2003). This approach however remains in the theoretical vein of discounted utility, which has been criticized in the sustainability literature and qualified as a "dictatorship of the present" (Chichilnisky, 1996).

If sustainability requires the sustaining of utility for intergenerational equity concerns, the *maximin* criterion (Solow, 1974; Cairns and Long, 2006) is a candidate to address this issue. This criterion emerges from the Rawls (1971) conception of justice and equity. It maximizes the utility of the poorest generation (or the minimal utility over time in a continuous time framework). The main result of this approach is that the maximin path is characterized by Hartwick's rule, which requires the investing of rents from exhaustible resources in reproducible capital to compensate for the depletion of their stocks (Hartwick, 1977). This rule has been generalized, that is to say that a nil net investment is required to keep the total productivity of all stocks constant, and sustain the consumption or utility (Dixit et al., 1980). This rule, related to the concept of genuine-saving, is argued to be a condition for sustainability (Withagen and Asheim, 1998; Mitra, 2002). However, Cairns (2008) emphasizes a limit of the maximin approach. As a maximin program is only defined for optimal economies, it cannot be expressed for a distorted economy. From that point of view, it is not straightforward to compute the sustainability indicators provided by the maximin approach for real economies. Martinet (2007) shows, in a particular case, that following Hartwick's investment rule sup-optimally (i.e., with prices that differ from the shadow values of the maximin problem) may lead to a decrease of the sustainability, measured by the value function of the maximin approach. An important theoretical challenge to address the sustainability issue is thus to extend the maximin approach to study the sustainability of economies that are not at the maximin optimum.

In this paper, we propose a framework to extend the maximin beyond optimality, which could be used to analyze the sustainability of economic trajectories which differ from the maximin path. This framework is based on the *viability* approach (Aubin, 1991) or weak-invariance approach (Clarke et al., 1995) which characterizes intertemporal dynamic trajectories regarding their consistency with given state and control constraints. Interpreting viability constraints as minimal rights to be guaranteed to all generations, the viability approach can be used to address the sustainability issue (Martinet and Doyen, 2007; Baumgärtner and Quaas, 2009). In all viability studies, the so-called *viability kernel* plays a major mathematical role. This set is the set of all initial (economic) states from which start viable (economic) trajectories, i.e., trajectories respecting the given constraints at all times, for example sustainability constraints. Therefore, using the viability approach, it is possible to define, in a given model, all the economic trajectories sustaining a specific, not necessarily maximal, utility level. From that point of view, the viability approach provides a relevant tool to study the sustainability of "sub-optimal economies" which differ from the maximin path.

We exhibit the strong links between *maximin* and *viability*. More specifically, we show that the value function of the maximin problem is the solution of a static optimization problem under constraints, involving the viability kernel. Our results are given in a general and abstract framework. Particular emphasis is put on the Hamiltonian formulation of the viability problem, that we interpret as a *weak* Hartwick rule. We relate this result to positive net investment *at maximin prices*, and describe how it makes it possible to characterize the sustainability of any development path. Our results are then illustrated for the canonical Dasgupta-Heal-Solow model (Dasgupta and Heal, 1974, 1979; Solow, 1974) often used to investigate sustainability issues.

The remainder of the paper is organized as follows. We first present in Section 2 the links between maximin and viability in terms of states and value functions. We then present in Section 3 how these links allow us to characterize maximin trajectories within the viability framework. In Section 4, we discuss the potential use of our framework, which extends maximin with viability, to examine the sustainability of trajectories which are not maximin paths. Our results are illustrated in the canonical Dasgupta-Heal-Solow model in section 5. We conclude in Section 6. The proofs of propositions lie in the appendix.

2 Maximin and Viability

2.1 A general dynamic economic model

Consider an economy with n capital stocks (either manufactured capital, labor or natural resources) and m economic decision parameters (consumption, investment or resource extraction). This economy is characterized by the state $X(t) \in \mathbb{R}^n_+$ and the control $u(t) \in \mathbb{R}^m$. All the economic dynamics are captured by a function $f : \mathbb{R}^n_+ \times \mathbb{R}^m \mapsto \mathbb{R}^n$ which may involve capital dynamics, production functions or natural resource growth functions. This economy is thus represented by the controlled dynamic system [1]

$$\dot{X}(t) = f(X(t), u(t)), \quad t \in \mathbb{R}_+. \tag{1}$$

At each time t, states and controls have to belong to some admissibility set represented by inequalities (e.g., positivity of consumption, irreversibility of investment, availability of labor, scarcity of resource) of the form:

$$g_i(X(t), u(t)) \geq 0, \qquad for \quad i = 1, \ldots, q. \tag{2}$$

Initial economic state at time $t_0 = 0$ is denoted $X(t_0) = X_0$. We shall denote by $X(\cdot)$ and $u(\cdot)$ state and control trajectories.

Consider the payoff function $L(X(t), u(t))$ which may depend on state and control. In economic terms, this payoff represents instantaneous utility.

2.2 The maximin approach

The maximin approach (Solow, 1974; Cairns and Long, 2006) aims at maximizing the minimal utility over time. In other words, the maximin criterion defines the maximal level of utility that can be sustained given economic endowments, i.e., from the initial economic state X_0. Hence, the maximin value function $V(.) : \mathbb{R}^n_+ \to \mathbb{R}$ is defined by

$$V(X_0) = \sup_{(X(\cdot),u(\cdot))} \left(\inf_{t \in \mathbb{R}_+} L(X(t), u(t)) \right) \tag{3}$$

$$s.t. \quad \begin{cases} \dot{X}(t) = f(X(t), u(t)) \ , \\ g_i(X(t), u(t)) \geq 0 \ , \\ X(t_0) = X_0 \ . \end{cases}$$

Whenever the supremum is reached and corresponds to a maximum, this criterion defines an optimal economic trajectory $X^\star(\cdot)$, with associated optimal economic decisions $u^\star(\cdot)$.

If the problem is *regular* (Burmeister and Hammond, 1977; Cairns and Long, 2006), the maximin path is associated with a constant utility over time, i.e.,

[1] We focus on time autonomous problems for the sake of exposition clarity.

$L(X^\star(t), u^\star(t)) = V(X_0)$, for all $t \in \mathbb{R}_+$. This can be interpreted as intergenerational equity (equality) from a sustainability point of view (Heal, 1998). However, this constant utility should not hide an important property of the maximin approach, that is to say that it is a dynamic approach. In fact, at any point of a regular maximin path, the utility is equal to the maximin value of the associated economic state, i.e., $L(X^\star(t), u^\star(t)) = V(X^\star(t))$. This is because the maximin path is regular, and thus the maximin value is constant along the path, that the consumption is constant.[2] In such a regular case, it turns out that the optimal controls satisfy Hartwick's rule, which, in such an abstract model, requires to keep the total value of net investment nil (Hartwick, 1977, 1978; Dixit et al., 1980; Mitra, 2002), allowing to compensate for the natural resources stock depletion by reproducible capital accumulation in order to maintain total productive capacities (Solow, 1986).[3]

2.3 The viability approach

The viability approach aims at studying the consistency between a dynamic system and a set of so-called *viability constraints*. It determines the conditions for these constraints to be satisfied at all times. These constraints can represent sustainability objectives to be achieved for all generations (Béné et al., 2001; Martinet and Doyen, 2007; Baumgärtner and Quaas, 2009). For example, it is possible to study the viability of an economy under a guaranteed utility constraint $L(X(t), u(t)) \geq L_{\min}$ to be satisfied over time.

The so-called *viability kernel* (Aubin, 1991) plays a major mathematical role in the viability analysis. It is the set of all states $X(t_0)$ such that from any of those states there are admissible decisions resulting in trajectories satisfying the given constraints at all times. Here we consider a specific viability kernel associated with dynamics (1), inequality constraints (2), and the following viability constraint requiring a guaranteed payoff $L_{\min}$:

$$L(X(t), u(t)) \geq L_{\min}. \tag{4}$$

We introduce the viability kernel, which depends on the guaranteed payoff level $L_{\min}$:

[2] For a clarifying description of the maximin approach, especially when the maximin path is non-regular, see Cairns and Tian (2010).
[3] If the problem is *non-regular*, the utility may increase or decrease over time along the maximin path (Asako, 1980; Cairns and Tian, 2010).

$$\mathrm{Viab}(L_{\min}) = \left\{ X_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists (X(\cdot), u(\cdot)) \text{ such that } \forall t \in \mathbb{R}_+, \\ \dot{X}(t) = f(X(t), u(t)), \\ X(t_0) = X_0, \\ g_i(X(t), u(t)) \geq 0, \quad for \quad i = 1, \ldots, q, \\ L(X(t), u(t)) \geq L_{\min}. \end{array} \right\} \tag{5}$$

From a general mathematical point of view, this set is a subset (potentially empty) of the state domain $\mathbb{R}^n_+$. The states belonging to the viability kernel have sufficient productivity capacities to sustain (at least) utility $L_{\min}$.

The viability kernel captures an irreversibility mechanism. From the very definition of this kernel, it is not possible to sustain $L_{\min}$ from any state lying outside the viability kernel; whatever are the decisions, trajectories leaving the viability kernel will eventually violate the constraints in finite time. It means that a necessary condition to sustain a given guaranteed utility level $L_{\min}$ is that the economic trajectory evolves within the associated viability kernel $\mathrm{Viab}(L_{\min})$. Viable trajectories are thus inward or tangential to the viability kernel (Aubin, 1991). We will use this important property in section 3.

2.4 Maximin as the optimization of viability

In this section, we characterize the maximin value function $V(.)$, defined by eq. (3), through a static optimization problem involving the viability kernel defined by eq. (5). It allows us to interpret maximin as an extreme case of viability.

We start from the following simple proposition.

Proposition 1 *Assume the existence of a maximin optimal solution* $(X^\star(\cdot), u^\star(\cdot))$ *starting from state* X_0 *at time* t_0*. Then*

$$X_0 \in \mathrm{Viab}(V(X_0)).$$

The proof of the proposition is presented in the appendix.

The interpretation of this simple proposition is that an economic endowment X_0 makes it possible to guarantee a utility level equal to the initial maximin value $V(x_0)$.

We now present the main proposition of the paper, which will be the corner stone of our key results.

Proposition 2 *For any initial conditions* (X_0)*, we have*

$$V(X_0) = \sup \Big(L_{\min} | \ X_0 \in \mathrm{Viab}(L_{\min}) \Big)$$

The proof of the proposition is detailed in the appendix.

We interpret this result as follows. We know that a utility level $L_{\min}$ is sustainable from initial state X_0 if X_0 belongs to the viability kernel $\mathrm{Viab}(L_{\min})$. The higher the utility level $L_{\min}$ to sustain, the less numerous the initial economic states making it possible to sustain it, i.e., the smaller the viability kernel $\mathrm{Viab}(L_{\min})$. The maximal sustainable utility (maximin value) will corresponds to the highest viability constraints for which the associated viability kernel contains initial state X_0. It also obviously means that, from a given X_0, no utility greater than the maximin value $V(X_0)$ can be sustained.

The interpretation of this proposition is quite simple but has powerful implications. It means that the maximin value can be defined within the viability framework using a static optimization problem on the viability kernel. Whenever the solution of a given optimization problem can be formulated in terms of viability kernel, the solution inherits the properties of the kernel. Such properties will allow us to shed a new light on the maximin, considering both the optimal trajectory and other paths.

3 Sustainability, optimality and sub-optimality

3.1 Maximin trajectory and viability kernels

We first show that the maximin trajectory evolves within the viability kernel associated with the constraint level $V(X_0)$. This result is valid whether the maximin path is regular or not.

Proposition 3 *Assume that there exists a maximin optimal solution* $(X^\star(\cdot), u^\star(\cdot))$ *starting from state* X_0*, then*

$$X^\star(t) \in \mathrm{Viab}(V(X_0)), \ \forall t \geq t_0 \ .$$

The proofs of the proposition is given in the appendix.

The maximin trajectory thus remains within the viability kernel associated with the constraint $L(X^\star(t), u^\star(t)) \geq V(X_0)$. In other words, the economic states along the maximin trajectory correspond to a sequence of combination of capital stocks making it possible to sustain the associated payoff objective $V(X_0)$ throughout time, maintaining the productive capacities of the economy. From the viability point of view, the maximin trajectory is viable, for the constraint $L(X(t), u(t)) \geq V(X_0)$. Consequently, it can be characterized using the properties of viable trajectories. In particular, the viable trajectories can either enter in the interior of the kernel $\mathrm{Viab}(V(X_0))$ or remain on the boundary of this kernel as illustrated by Fig. 1. Several examples, and especially the canonical DHS example addressed in section 5 of this paper, suggest that for regular maximin problems, the regular maximin paths are sticked to the boundary of the viability kernel. For regular maximin paths, the maximin trajectory evolves among states which have the same maximin value. These particular economic states have been interpreted as *capital valuation contours* by Burmeister and Hammond (1977). In other words, these states have the same long-run productivity potential. Thus the boundary of the viability kernel is deeply related to that *capital valuation contour*. By contrast, a non-regular maximin path is inward to the viability kernel, and has a non-decreasing maximin value. [4]

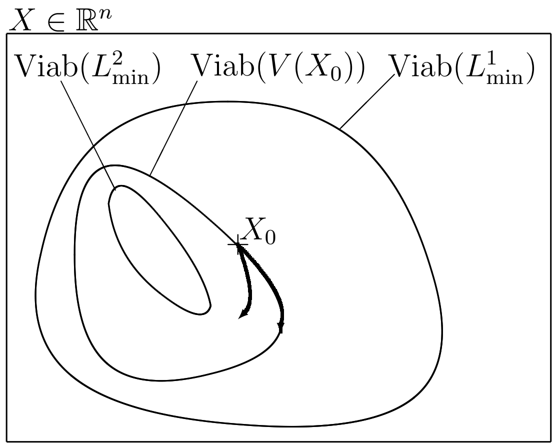

Fig. 1. Viability kernels and maximin trajectories. $L^2_{\min} > V(X_0) > L^1_{\min}$. The higher the viability constraint, the smaller the viability kernel. The maximin trajectory evolves along the boundary of the kernel (tangential trajectory) or can be inward.

[4] Either the maximin value is constant for non-regular paths characterized by "globally bounded" maximin value (such as the simple fishery or the Ramsey model in which fish or capital stocks above the MSY or golden rule value are redundant and not contributing to the maximin value, as described by Asako (1980)), or the maximin value is increasing along with the utility for non-regular path characterized by "locally bounded" utility as described by Cairns and Tian (2010).

Analyzing previous ideas from a marginal viewpoint, these results have sound connections with the basic idea underlying Hartwick's investment rule to maintain the total capital productivity constant (Hartwick, 1977; Dixit et al., 1980; Solow, 1993), and more generally with genuine-saving. We now explore this link.

3.2 Hamiltonian formulation of the viability problem: a weak Hartwick rule

Viable trajectories are inward or tangential to the viability kernel (Aubin, 1991). It is possible to characterize these trajectories within the viability framework using a Hamiltonian formulation and normal cones. For this purpose, we introduce the following Hamiltonian:

$$H(X, u, p) =< p, f(X, u) >= \sum_{i=1}^{n} p_i f_i(X, u).$$

Following Aubin (1991), it turns out that the *viable decisions* $u(t)$ associated with a viable trajectory starting from $X(t) \in \mathrm{Viab}(L_{\min})$ are solution of a specific Hamilton-Jacobi-Bellman inequality:

$$H(X(t), u(t), p(t)) \geq 0, \quad \forall p(t) \in N_{\mathrm{Viab}(L_{\min})}(X(t)) \ , \tag{6}$$

with $N_{\mathrm{Viab}(L_{\min})}(X)$ the normal cone to set Viab at state X (see for instance Aubin and Frankowska (1990); Rockafellar and Wets (1998)).

When the guaranteed utility threshold $L_{\min}$ coincides with the maximin value, namely $L_{\min} = V(X)$, these normal cones are directly connected to the marginal value $V_x(X)$ of the maximin value function, and we can derive the following Hamilton-Jacobi condition for the maximin value function V.

Proposition 4 *Assume that the dynamic system (f, g, L) is smooth enough.*[5] *Then the maximin value function V is a solution of the following Hamilton Jacobi inequality for any X such that $V(X) < +\infty$:*

$$\sup_{u \in A(x)} H(X, u, V_x(X)) \geq 0, \tag{7}$$

where $A(x) = \{u, \ L(X, u) \geq V(x), \ g_i(X, u) \geq 0\}$ and V_x means the derivative[6] *of V. Moreover, optimal controls are solutions of the Hamilton-Jacobi-*

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[5] For instance, if $U(x) = \{u, \ L(X, u) \geq L_{\min}, \ g_i(X, u) \geq 0\}$ is convex, closed and bounded together with f, g and L continuous.

[6] It could be the derivative in a very weak sense with contingent derivative. See Aubin and Frankowska (1990); Rockafellar and Wets (1998); the (upper) contingent

Bellman inequality

$$u^\star(X) \in \operatorname{Arg}\max_{u\in A(x)} H(X,u,V_x(X)). \tag{8}$$

The proposition is proved in the appendix.

We interpret this Hamiltonian condition as a *weak* Hartwick rule: Consider any maximin trajectory $X^\star(.)$ associated with optimal control $u^\star(.)$. From previous Hamiltonian assertions, assuming that V is smooth enough, we deduce that

$$\begin{aligned} \tfrac{d}{dt}V(X^\star(t)) &= < V_x(X^\star(t)), \tfrac{d}{dt}X^\star(t) > \\ &= < V_x(X^\star(t)), f(X^\star(t),u^\star(t)) > \\ &= H(X^\star(t),u^\star(t),V_x(X^\star(t))) \\ &\geq 0 \end{aligned} \tag{9}$$

Thus, along any maximin path $X^\star(t)$, the time derivative of the maximin value function, representing the total capital productivity, remains positive. [7] In other words, the total productivity is non-decreasing, which corresponds to some genuine-saving, with our weak version of Hartwick's rule. This correspond to a viable trajectory.

Of interest is the fact that such a rule is obtained without strong regularity assumptions on the maximin value function (upper semi-continuous value function is enough) which can occur if other constraints g are binding. In particular, our result holds true whether the maximin path is regular or not.

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derivative $f_x(x)(v)$ of a function $f:\mathbb{R}^n\to\mathbb{R}$ in the direction v is defined by

$$f_x(x)(v) = \limsup_{h\to 0^+,\, v'\to v} \frac{f(x+hv')-f(x)}{h}.$$

Such contingent derivatives coincide with usual derivatives when the function f is smooth enough. But they can be defined for only semi-continuous function which can often occur for value function of optimal control under constraints.

[7] The maximin value stays constant along regular maximin path. It is also locally constant along non-regular path characterized by redundant stocks (Asako, 1980), with $V_x(X)=0$. The derivative may be strictly positive along non-regular maximin path such as in Cairns and Tian (2010) when the capital accumulation increases the maximin value.

4 Characterizing the (un)sustainability of economic development paths

We have shown that the maximin path of a problem and more generally the maximin value function can be characterized using the viability approach, by maximizing the viability constraint associated with the payoff. We argue that the viability approach makes it possible to study the sustainability of trajectories which differ from the maximin path. These trajectories are sub-optimal with respect to the maximin criterion in the sense that the sustained level of utility is lower than the maximin value. There are several reasons for a given trajectory to deviate from the maximin path. The most obvious one is when maximin is not the criterion defining the economic development path. Another particular case of such sub-optimal economy is studied by Martinet (2007) who examined how the maximin value function evolves along a constant consumption path which follows the Hartwick investment rule with sub-optimal prices.

We here consider the Weak Sustainability paradigm à la Solow, in the sense that "sustainability ... must amount to an injunction to preserve the productive capacity for the indefinite future" (Solow, 1993, p.163) and that "a sustainable path ... is one that *allows* every future generation *the option* of being as well off as its predecessors" (ibid. p.168 ; our emphasize). We thus do not assume that utility *must* be non-decreasing, and focus on the possibility to sustain utility. For a comprehensive discussion of sustainability paradigms, see Neumayer (2010).

We shall consider two levels of knowledge of the trajectory under study. First we quickly examine the case in which the whole trajectory is known. Second, we consider the more interesting case in which only the current economic state and decisions are known.

Sustainability of a path: Consider a whole economic trajectory $(X(t), u(t))$ defined by arbitrary open-loop controls or by a given feedback rule as in Arrow et al. (2003) and Vouvaki and Xepapadeas (2008). Assume that it deviates from the maximin path in the sense that

$$\exists\, t^* \text{ such that } L(X(t^*), u(t^*)) < V(X(t^*)).$$

This trajectory does not sustain the maximin level of utility at time t^*. An easy way to characterize the sustainability of this economy is to consider the minimal value of utility over time, i.e., the utility level which is actually sustained

$$L_{\min} = \min_{t \geq 0} L(X(t), u(t)).$$

In the viability framework, such a trajectory is characterized by the viability kernel $\mathrm{Viab}(L_{\min})$ associated with this sustained level of utility. This sustained level is sub-optimal with respect to the maximin criterion but well characterized in the viability framework.

Sustainability of the current state and decision: If the whole economic trajectory is unknown, our framework still allows us to assess the sustainability of current decisions u_0. To know if current utility $L(X_0, u_0)$ is sustainable, we can examine the location of the economic state X_0 with respect to the viability kernel $\mathrm{Viab}(L(X_0, u_0))$. This provides a first order condition for sustainability:

$$\textit{First order condition for sustainability:} \;\; X_0 \in \mathrm{Viab}(L(X_0, u_0)). \tag{10}$$

However, even in the favorable case where X_0 belongs to $\mathrm{Viab}(L(X_0, u_0))$, a marginal condition relying on the Hamiltonian condition (7) is relevant to characterize the sustainability of the economy, and in particular of economic decisions on investment. This provides a second order condition for sustainability:

$$\textit{Second order condition for sustainability:} \;\; H(X_0, u_0, V_x(X_0)) \geq 0. \tag{11}$$

The intuition underlying such sustainability characterization is that the function $t \rightarrow V(x(t))$ is locally increasing at time t_0 as in condition (9) or equivalently that the (weak) Hartwick's rule holds true.

In such a framework, we can discuss the three following cases depending on current utility levels $L(X_0, u_0)$:

- $L(X_0, u_0) > V(X_0)$. In this case, the current utility is higher than the maximin value. This case has been discussed in the literature and qualified as unsustainability by Pezzey (1997) [8]. From proposition 2, we deduce that $X_0 \notin \mathrm{Viab}(L(X_0, u_0))$. In other words, the economy is faced with unsustainability of first kind: the current economic state does not make it possible to sustain the current consumption. In particular, let us point out that such unsustainability situation can also occur for some non-regular (optimal) maximin paths where the utility $L(X_0, u_0^\star)$ may be greater than the maximin value for some time. This is the case for example for the simple

[8] Pezzey (1997) describes three different constraints (using our notations, but Pezzey's terminology)

(1) If $L(X(t), u(t)) \leq V(t, X(t))$, $\forall t$, development is said to be *sustainable.*
(2) If $\frac{L(X(t),u(t))}{dt} \geq 0$, $\forall t$, development is said to be *sustained.*
(3) If $L(X(t), u(t)) \geq L_{\min}$, development is said to be *survivable.* This last constraint corresponds to a viability constraint.

fishery economy, if the fish stock is larger than the Maximum Sustainable Yield Stock, or for the simple Ramsey model. In this case, the utility will eventually decrease to the maximin value after some finite time.

- $L(X_0, u_0) = V(X_0)$. In this case, the current utility is equal to the maximin value and consequently, the first condition for sustainability is satisfied since, from proposition 1, $X_0 \in \text{Viab}(V(X_0)) = \text{Viab}(L(X_0, u_0))$. However, decisions u_0 may not correspond to some optimal maximin feedback $u^\star(X_0)$. If our weak Hartwick rule is not satisfied, the second condition for sustainability does not hold. In such a negative case, the total productivity of the economy (the maximin value) locally decreases and the trajectory leaves the viability kernel $\text{Viab}(V(X_0))$.
- $L(X_0, u_0) < V(X_0)$. In this case, the first condition for sustainability is satisfied as $X_0 \in \text{Viab}(V(X_0)) \subset \text{Viab}(L(X_0, u_0))$, and current utility can be sustained as it is lower than the maximin value. Nevertheless, depending on current decision u_0, the trajectory may either enter within the viability kernel or leave it. The trajectory is thus either sustainable or non-sustainable depending on investment. In the former case, the maximin value increases and corresponds to a non-negative net investment at maximin prices, satisfying our second condition for sustainability. In the latter case, net investment is negative and the maximin value decreases.

The last two cases advocate the accommodation of both the maximin value and viability kernels as indicators of sustainability. It is worthy to note that Pezzey's condition for *sustainability* (having a current utility lower than the maximin level) is in fact necessary but not sufficient to obtain our characterization of sustainability. Based on the results of this paper, we argue that studying the sustainability of the economy in the viability framework is equivalent to consider that a path is sustainable if simultaneously its current payoff or utility is sustainable in the sense of Pezzey (first order condition for sustainability) and our weak Hartwick rule applies (second order condition for sustainability) such that the maximin value is increasing at the current time. This characterizes both the current state, decision and dynamics of the economy. In particular, note that the maximin value can decrease even if utility is lower than its level. In such a context, what matters is the time derivative of the maximin value instead of that of the discounted utility. The difference is very important in terms of accounting. Unsustainability occurs if genuine saving *at maximin prices* (shadow values) is negative. Sustainability accounting should thus be based on maximin prices which are related to the long-run productivity and sustainability of the economy rather than on discounted utility prices which are related to the discounted, short-run consumption.

5 The Dasgupta-Heal-Solow model

5.1 A consumption-production economy with a non-renewable resource

To illustrate the results of previous section, we study a canonical model often used to address the sustainability issue in exhaustible resource economics: the Dasgupta-Heal-Solow model (Dasgupta and Heal, 1974, 1979; Solow, 1974; Heal, 1998). This is an intertemporal resource allocation model with a manufactured capital stock $K(t)$ and non renewable natural resources $S(t)$. The rate of extraction of natural resources is $r(t)$. Natural resources and capital are used to produce a composite good with a Cobb-Douglas technology represented by the production function $f(K,r) = K^{\alpha}r^{\beta}$, with $\beta < \alpha \leq 1$. The production can either be invested to accumulate capital $\dot{K}$ or consumed $c(t)$.

The economy, represented by state (K, S) and control (c, r) is subject to dynamics

$$\begin{cases} \dot{K}(t) = K(t)^{\alpha}r(t)^{\beta} - c(t), \\ \dot{S}(t) = -r(t). \end{cases} \tag{12}$$

We assume that both natural resources and capital stocks must remain non-negative, that the extraction $r(t)$ is irreversible, and that consumption cannot exceed the production level (investment is irreversible), which imply the following admissibility constraints:

$$\begin{cases} 0 \leq K(t), \\ 0 \leq S(t), \\ 0 \leq r(t), \\ 0 \leq K(t)^{\alpha}r(t)^{\beta} - c(t). \end{cases} \tag{13}$$

5.2 The maximin approach

This model has been studied within the maximin framework by Solow (1974) and Dasgupta and Heal (1979). The purpose was to determine the maximal sustainable consumption in an economy with an essential non-renewable resource. This objective reflects an intergenerational equity concern. [9] The problem reads

[9] The problem could be stated in utilitarian terms, assuming an increasing utility function, without modifying the results. However, to be consistent with the Solow's analysis, we assume that the objective is to sustain the consumption level.

$$V(S_0, K_0) = \max_{\begin{cases} c(\cdot), r(\cdot) \\ \text{satisfying } (12), (13) \\ S(t_0) = S_0,\ K(t_0) = K_0 \end{cases}} \min_{t \in \mathbb{R}_+} c(t) \tag{14}$$

According to Solow's result (Solow, 1974, p 39), the maximal sustainable consumption in this model, which is also the maximin value function for the initial state (K_0, S_0), is

$$V(S_0, K_0) = (1-\beta)\Big(S_0(\alpha-\beta)\Big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}}. \tag{15}$$

The associated maximin path is regular, and is characterized by a constant consumption $c^\star(t) = V(S_0, K_0)$. The path evolves through states which have the same "maximin value," and these states constitute what Burmeister and Hammond (1977) name a capital valuation contour. At each time, one has $c^\star(t) = V(K^\star(t), S^\star(t)) = V(S_0, K_0)$. It is made possible only because the depletion of the resource stock (or more precisely the decreasing extraction and use of the natural resource) is compensated for by the capital accumulation at an adequate level, which is defined by the Hartwick rule (Hartwick, 1977). In this model, the Hartwick rule reads $\dot{K} = r f'_r = \beta K^\alpha r^\beta$. A part β of the production is invested. The consumption is a part $(1-\beta)$ of the production, and is also constant. The production level is constant. [10]

In this model, the maximin path is unique, efficient and, of course, optimal with respect to the maximin criterion. It maximizes the sustained level of consumption.

5.3 *The viability approach*

The sustainability of the DHS model has been studied within the viability framework by Martinet and Doyen (2007). To address the sustainability issue in this model, the viability approach is based on a guaranteed consumption level $c_{\min}$, that has to be sustained over time:

$$0 < c_{\min} \leq c(t). \tag{16}$$

In the present case, the viability kernel corresponds to the set

[10] See Solow (1974), Hartwick (1977), Dasgupta and Heal (1979) and Cairns and Long (2006) for details of the maximin solution in this model.

$$\mathrm{Viab}(c_{\min}) = \left\{ (S_0, K_0) \;\middle|\; \begin{array}{l} \exists \text{ decisions } (c(\cdot), r(\cdot)) \text{ and states } (S(\cdot), K(\cdot)) \\ \text{starting from } (S_0, K_0) \text{ satisfying dynamics (12)} \\ \text{and constraints (13) and (16) at any time } t \in \mathbb{R}_+ \end{array} \right\}.$$

The viability kernel of this problem is given by the following expression (Martinet and Doyen, 2007, Proposition 3).

$$\mathrm{Viab}(c_{\min}) = \Big\{ (S, K) \text{ such that } S \geq \mathcal{S}(K, c_{\min}) \Big\}$$

where $\mathcal{S} : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a function defined by

$$\mathcal{S}(K, c_{\min}) = \frac{1}{\alpha - \beta} \left(\frac{c_{\min}}{1 - \beta} \right)^{\frac{1-\beta}{\beta}} K^{\frac{\beta-\alpha}{\beta}}. \tag{17}$$

According to eq.(17), a sustainability condition linking resource and capital stocks is required. Resource stock S has to be larger than a threshold $\mathcal{S}(K, c_{\min})$ depending on capital stock K and sustainability objective $c_{\min}$. The higher the capital stock (*ceteris paribus*), the lower this threshold. The higher the sustainability objective (*ceteris paribus*), the higher this threshold. From the very definition of the viability kernel, if the initial state is not within the viability kernel, it is not possible to sustain the consumption level $c_{\min}$ over an infinite time horizon. On the contrary, from any state within the viability kernel, it is possible to sustain $c_{\min}$. In other words, it is possible to sustain a given level of consumption $c_{\min}$ only by staying within the associated viability kernel $\mathrm{Viab}(c_{\min})$. The viability approach thus defines the conditions to sustain a consumption level, without maximizing this level. As a consequence, there is not necessarily a unique sustainable path starting from a given initial state. Any path satisfying the viability constraint (sustaining the minimal consumption level) is viable. Relevant decisions consist in maintaining the state within the viability kernel. From any state which is strictly within the viability kernel (not on the border), every admissible control is relevant, i.e., viable decisions must belong to the set $\mathcal{C}(K, S) = \Big\{ (r, c) \Big| c_{\min} \leq c \leq K^\alpha r^\beta \Big\}$. In particular, sustainable paths are not reduced to constant consumption paths, and the extraction does not necessarily satisfy the Hartwick rule (Martinet and Doyen, 2007). However, on the boundary of the viability kernel, i.e., if $S = \mathcal{S}(K, c_{\min})$, a specific path must be followed to ensure that the velocities $(\dot{K}, \dot{S})$ are tangent to the viability kernel. Applying the Hamiltonian characterization (7), the viable feedbacks $u^* = (r^*, c^*)$ are the solution of the following Hamilton-

Jacobi-Bellman inequality

$$\begin{cases} \max\limits_{(r,c)} & H(S,K,r,c,p_1,p_2) \geq 0 \\ K^\alpha r^\beta \geq c \geq c_{\min} & \\ r \geq 0 & \end{cases}$$

with the Hamiltonian defined by

$$H(S,K,r,c,p_1,p_2) = -p_1 r + p_2(K^\alpha r^\beta - c).$$

Specific computations detailed in Martinet and Doyen (2007) implies that viable feedback decisions are reduced to

$$r^*(S,K) = \left(\frac{c_{\min}}{1-\beta}\right)^{\frac{1}{\beta}} K^{-\frac{\alpha}{\beta}} \text{ and } c^*(S,K) = c_{\min}.$$

The consumption then remains constant and the Hartwick investment rule holds true. [11]

5.4 Maximin as the optimization of viability

To present the link between the maximin framework and the viability approach in the particular DHS model, we show that proposition 2 is satisfied in this illustrative model. For this purpose, we compute the maximum viability constraint $c_{\min}$ for which a given initial state (K_0, S_0) still belongs to the associated viability kernel $\mathrm{Viab}(c_{\min})$. We denote this level $c^+_{\min}(K_0, S_0)$:

$$c^+_{\min}(K_0,S_0) = \max\left(c_{\min} \;\middle|\; (K_0,S_0) \in \mathrm{Viab}(c_{\min})\right)$$

This "maximum viability constraint" solves the equation

$$\mathcal{S}(K_0, c^+_{\min}) = S_0$$

where $\mathcal{S}(K_0, c_{\min})$ is given by the equation (17). We obtain

$$c^+_{\min} = (1-\beta)\Big(S_0(\alpha-\beta)\Big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}} \tag{18}$$

which means, according to eq.(15), $c^+_{\min} = V(S_0, K_0)$.

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[11] We have $r^* = \left(\frac{c^*}{1-\beta}\right)^{\frac{1}{\beta}} K^{-\frac{\alpha}{\beta}}$ or equivalently $c^* = (1-\beta)K^\alpha r^{*\beta}$ which is also the Hartwick investment rule, i.e., $\dot{K} = \beta K^\alpha r^\beta = r f'_r$.

The boundary of the viability kernel $\mathrm{Viab}(c^+_{\min}(K_0, S_0))$ thus represents the capital valuation contour associated with the maximin path starting from (K_0, S_0).

5.5 Graphical illustration

To illustrate our argument, we use a graphical representation of our result. Fig. 2 represents three viability kernels ($\mathrm{Viab}(\bar{c}) \subset \mathrm{Viab}(V(K_0, S_0)) \subset \mathrm{Viab}(\underline{c})$, with $\underline{c} < V(K_0, S_0) < \bar{c}$) and five trajectories starting from a given initial state (K_0, S_0): the maximin trajectory and four trajectories illustrating the three cases described in section 4, page 12.

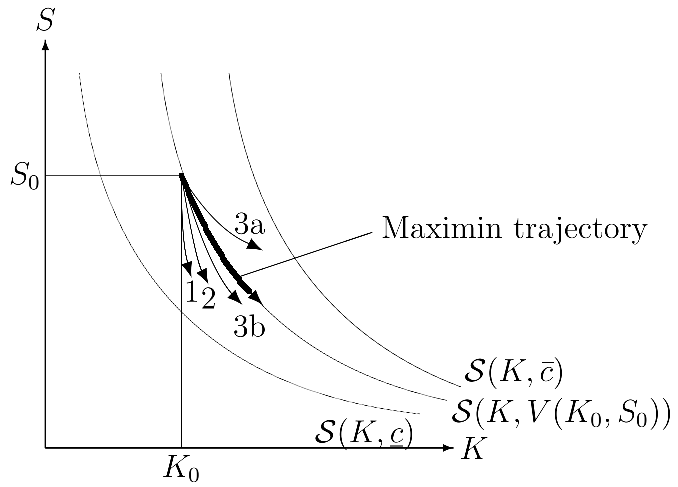

Fig. 2. Viability kernels ($\mathrm{Viab}(\bar{c}) \subset \mathrm{Viab}(V(K_0, S_0)) \subset \mathrm{Viab}(\underline{c})$, with $\bar{c} > V(K_0, S_0) > \underline{c}$) are respectively the Epigraph of curves $\mathcal{S}(K, \bar{c})$, $\mathcal{S}(K, V(K_0, S_0))$, and $\mathcal{S}(K, \underline{c})$. The Maximin trajectory and four sub-optimal trajectories are represented. Trajectories 1, 2 and 3b are unsustainable. The maximin trajectory and trajectory 3a are sustainable.

The various trajectories are interpreted as follows:

Maximin: The maximin trajectory follows the boundary of the viability kernel associated to the viability constraint $c(t) \geq c^+(K_0, S_0) = V(K_0, S_0)$, i.e., $\mathcal{S}(K, c^+(K_0, S_0))$, and thus sustains the maximin value.

Traj 1: Trajectory 1 is characterized by a consumption larger than the maximin value, $c_0 = \bar{c} > V(K_0, S_0)$; The economy is faced with the first kind of unsustainability: the current state is not within the viability kernel of current consumption, i.e., $(K_0, S_0) \notin \mathrm{Viab}(\bar{c})$. Moreover, the trajectory is leaving $\mathrm{Viab}(V(K_0, S_0))$. The maximin value decreases as genuine saving is negative.

Traj 2: Trajectory 2 has a consumption equal to the maximin level, $c_0 = V(K_0, S_0)$, but extraction and investment are different from the maximin decisions. The second order condition for sustainability is not satisfied. The trajectory leaves the viability kernel $\mathrm{Viab}(V(K_0, S_0))$, which means that net investment at maximin prices is negative and the maximin value decreases.

Traj 3: The two remaining trajectories (3a and 3b) have a consumption lower than the maximin value, $c_0 = \underline{c} < V(K_0, S_0)$. These two trajectories satisfy the first order condition for sustainability as $(K_0, S_0) \in \mathrm{Viab}(\underline{c})$. Extraction and investment differ between the two trajectories:

3a: Trajectory 3a has higher investment, resulting in a positive net investment at maximin shadow values. The trajectory is entering the viability kernel $\mathrm{Viab}(V(K_0, S_0))$, which means that the maximin value increases. This trajectory thus also satisfies the second order condition for sustainability.

3b: Trajectory 3b is investing less. Net investment at maximin shadow values is negative, i.e., $H(S_0, K_0, r_0, c_0, V_K, V_S) < 0$ and weak Hartwick's rule is violated. The trajectory is leaving the viability kernel $\mathrm{Viab}(V(K_0, S_0))$. Maximin value decreases. This trajectory does not satisfy the second order condition for sustainability.

6 Conclusion

The maximin criterion defines the maximal utility level that can be sustained in an intergenerational equity perspective. Nevertheless, a maximin approach can be applied only to optimal economies, which makes the sustainability indicators it provides impossible to compute apart from the maximin path. An important challenge to address the sustainability issue in real economies is thus to extend the maximin approach beyond optimality to study the sustainability of economic trajectories which differ from the maximin path.

In this paper, we proposed to extend the maximin approach using the viability approach. The viability approach studies the consistency between a dynamic system and given constraints. It makes it possible, for instance, to define all the economic trajectories sustaining a given, not necessarily maximal, utility level. We exhibited the strong links between the *maximin* criterion and the *Viability* approach. In particular, we showed that the value function of the maximin problem is the solution of a static optimization problem under constraints, involving the so-called viability kernel defined in the viability approach.

On the one hand, our results emphasize the relevance of the viability approach to address the sustainability issue and deal with intergenerational equity concerns. In particular, viability defines decisions that satisfy the sustainability constraint now, and maintain the capability of the economy to satisfy this constraint in the future. From that point of view, the viability approach is consistent with the Brundtland definition of sustainability characterizing a sustainable development as a development “that meets the needs of the present without compromising the ability of future generations to meet their own needs.”

On the other hand, our results point out that extending the maximin approach with the viability approach provides a framework to study the sustainability of any (sub-optimal) economic trajectory. Taking the maximin as an objective often results in a constant utility, which has been criticized as perennial poverty may be an optimum. We argue in this conclusion that even if maximin is not taken as an objective, the maximin value function can be used as an indicator of sustainability along any trajectory, including sub-optimal ones. The viability approach then offers a framework to study the sustainability of development paths, characterized by the maintain of productive capacities.

Our result opens interesting research opportunities. The more challenging one may be to define supporting prices in the viability framework in practice. These prices would correspond to the maximin prices and have to be used in the computation of net investment to determine if the productive capacities of the economy are increasing or not.

A Proofs

A.1 Proof of proposition 1

Assume the existence of a maximin optimal solution $(X^\star(\cdot), u^\star(\cdot))$ starting from state X_0 at time t_0. Then

$$V(X_0) = \inf_{t \geq t_0} L(X^\star(t), u^\star(t))$$

Consequently

$$V(X_0) \leq L(X^\star(t), u^\star(t)), \ \forall t \geq t_0$$

In other words, state X_0 belongs to the viability kernel $\mathrm{Viab}(V(X_0))$.

A.2 Proof of Proposition 2

We proceed in two steps. First, consider the initial time t_0 and the initial state X_0. Pick up some $L_{\min}$ such that $X_0 \in \mathrm{Viab}(L_{\min})$. From the very definition of the viability kernel, this implies the existence of a feasible path $(\widehat{X}(t), \widehat{u}(t))$

such that

$$\begin{cases} \dot{\widehat{X}}(t) = f(\widehat{X}(t), \widehat{u}(t)) \\ \widehat{X}(t_0) = X_0 \\ g_i(\widehat{X}(t), \widehat{u}(t)) \leq 0 \qquad for \quad i = 1, \ldots, q \\ L(\widehat{X}(t), \widehat{u}(t)) \geq L_{\min} \end{cases} \tag{A.1}$$

We thus have $\inf_t L(\widehat{X}(t), \widehat{u}(t)) \geq L_{\min}$ and we deduce

$$V(X_0) = \sup_{(X(t),u(t)) \text{ satisfying } (A.1)} \inf_t L(X(t), u(t)) \geq L_{\min}$$

Since the inequality holds for any $L_{\min}$ such that $X_0 \in \mathrm{Viab}(L_{\min})$, this leads to

$$V(X_0) \geq \sup \Big(L_{\min} |\ X_0 \in \mathrm{Viab}(L_{\min}) \Big).$$

Conversely, from the definition of the value function $V(X_0)$, for any $n \in \mathbb{N}$, there exists an admissible (satisfying (A.1)) and maximizing sequence $(X_n^\star(\cdot), u_n^\star(\cdot))$ in the sense that

$$V(X_0) \geq \inf_t L(X_n^\star(t), u_n^\star(t)) \geq V(X_0) - \frac{1}{n}.$$

This implies $X_0 \in \mathrm{Viab}\Big(t_0, V(X_0) - \frac{1}{n}\Big)$ which leads to

$$V(X_0) - \frac{1}{n} \leq \sup \Big(L_{\min} | X_0 \in \mathrm{Viab}(L_{\min}) \Big)$$

and finally, letting n converges toward $+\infty$

$$V(X_0) \leq \sup \Big(L_{\min} |\ X_0 \in \mathrm{Viab}(L_{\min}) \Big).$$

Hence the equality holds true.

A.3 Proof of proposition 3

Assume an autonomous system where dynamics f, constraints g, and utility L do not depend on time t. Assume the existence of a maximin optimal solution $(X^\star(\cdot), u^\star(\cdot))$ starting from state X_0 at time t_0. Consider the translated path

$$\widetilde{X}(s) = X^\star(s - t_0 + t),\ \widetilde{u}(s) = u^\star(s - t_0 + t),\ s \geq t_0.$$

It is straightforward to prove that for any $s \geq t_0$

$$\begin{cases} \dot{\widetilde{X}}(s) = f(\widetilde{X}(s), \widetilde{u}(s)), \ \widetilde{X}(t_0) = X^\star(t) \\ g_i(\widetilde{X}(s), \widetilde{u}(s)) \geq 0 \\ L(\widetilde{X}(s), \widetilde{u}(s)) \geq V(X_0). \end{cases}$$

Therefore, $X^\star(t)$ belongs to the viability kernel $\mathrm{Viab}(V(X_0))$.

A.4 Proof of proposition 4

We first prove that the hypograph $\mathrm{Hyp}(V)$ of V is viable for the augmented dynamics:

$$\dot{X} = f(X,u), \ \dot{y} = 0, \ L(X,u) \geq y, \ g_i(X,u) \geq 0$$

where $\mathrm{Hyp}(V) = \{(X,y), \ y \leq V(X)\}$.

Pick up some $(X,y) \in \mathrm{Hyp}(V)$. By proposition 1 and existence assumptions due to the regularity of f, g and L, we deduce that $X \in \mathrm{Viab}(V(X))$. Thus there exists a solution $(X(\cdot), u(\cdot))$ such that for any $s \geq t$

$$L(X(s), u(s)) \geq V(X), \ g(X(s), u(s)) \geq 0, \ X(t) = X$$

Consider $y(t) = y$. Since $y \leq V(X)$, the augmented path $(X(\cdot), y(\cdot), u(\cdot))$ is also a solution of

$$\dot{X}(t) = f(X(t), u(t)), \ \dot{y} = 0, \ L(X(t), u(t)) \geq y(t), \ g_i(X(t), u(t)) \geq 0$$

starting from (X,y) at time t. This implies in particular that, for any time $t \geq t_0$, we have $(X(t), y(t)) \in \mathrm{Hyp}(V)$. In other words, the hypograph $\mathrm{Hyp}(V)$ is viable for the augmented dynamics.

In a second step, we use a variational characterization for the viability of $\mathrm{Hyp}(V)$. From the viability theorem in Aubin (1991) relying on inward tangential conditions, we deduce that

$$\forall (X,y) \in \mathrm{Hyp}(V), \ \exists\, u \text{ s.t. } L(X,u) \geq y, \ g_i(X,u) \geq 0,$$
$$(f(X,u), 0) \in T_{\mathrm{Hyp}(V)}(X,y)$$

where $T_{\mathrm{Hyp}(V)}(X,y)$ stands for the tangent cone of the set $\mathrm{Hyp}(V)$ at point (X,y). Such assertion is really informative on the boundary of $\mathrm{Hyp}(V)$ where $y = V(X)$. Moreover, from Aubin and Frankowska (1990) and Rockafellar and Wets (1998), we know that

$$T_{\mathrm{Hyp}(V)}(X, V(X)) = \mathrm{Hyp}\left(V_X(X)\right).$$

where $V_X(X)$ means the contingent derivative of V at X. We deduce that

$$\forall X \in \mathrm{Dom}(V), \exists\, u \text{ s.t. } L(X,u) \geq V(X),\ g(X,u) \geq 0,$$
$$V_X(f(X,u)) \geq 0$$

Using the Hamiltonian $H(X,u,p) =< p, f(X,u) >$, we conclude to obtain

$$\forall X \in \mathrm{Dom}(V), \qquad \sup_{u \in A(x)} H\Big(X, u, V_X(X)\Big) \geq 0. \tag{A.2}$$

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