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Document de Travail
Working Paper
2012-14

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March 26, 2012

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Abstract

We propose a new criterion which reflects both the concern for welfare (utility) and the concern for rights in the evaluation of economic development paths. The concern for rights is captured by a pre-ordering over combinations of thresholds (floors or ceilings on various quantitative indicators) that serve as constraints on actions and on levels of state variables. These thresholds are interpreted as minimal rights to be guaranteed to all generations. They are endogenously chosen within the set of all feasible thresholds, accounting for the "cost in terms of welfare" of achieving these rights. We apply the criterion to several examples, including the standard Dasgupta-Heal-Solow model of resource extraction and capital accumulation. We show that if the weight given to rights in the criterion is sufficiently high, the optimal solution may be on the threshold possibility frontier. The development path is then "driven" by the rights. In particular, if a minimal consumption is considered as a right, constant consumption can be optimal even with a positive utility discount rate. The shadow prices of thresholds play an important role in the determination of the rate of discount to be applied to social investment projects.

JEL-Classifications: D63, H43, Q01.

Keywords: Rights, Intergenerational Equity, Welfare, Sustainability.

1 Introduction

Much of normative economic theory is built on the premises that individuals seek to maximize their "utility" or "welfare,"and that social welfare is the sum (or weighted sum) of individual welfare. Under utilitarianism, or more generally welfarism, it is legitimate to prescribe policies that lead to increase in the welfare of some individuals at the expense of the welfare of other individuals, as long as "social welfare" rises. At some extreme, the life of a person could be sacrificed for "the greater good" of the society. In an intergenerational context, the welfare of a generation can be sacrificed without limit to increase the intertemporal welfare by raising the welfare of other generations. Many philosophers have expressed the concern that utilitarianism, or more generally welfarism, does not take "rights" seriously. They argue that all individuals should be entitled with some basic rights, such as life, health, and a "decent standard of living." John Rawls [23] pointed out that "optimal growth" (under some utilitarian objective) may unreasonably require too much savings from poor generations for the benefits of their wealthier descendants.[1] More recently, the same rationale has led environmentalists to argue that the present generations, in their pursuit of wealth and wellbeing, are depriving future generations of their rights to natural assets.

Sustainable development has been described in the Brundtland report [34] as development "that meets the needs of the present without compromising the ability of future generations to meet their own needs." Current patterns of growth, however, induce concerns for sustainability, and in particular with respect to environmental degradations. Intergenerational equity and environmental concerns are thus cornerstones of sustainability. Reflecting the concerns for rights, environmental issues are often addressed with quantitative approaches on physical measures, and thresholds. Along these lines, it is argued that society should impose constraints, in the form of floors or ceilings, on various

[1] In a similar vein, Immanuel Kant (1724-1804) found it disconcerting that earlier generations should carry the burdens for the benefits of later generations. In his essay, "Idea for a Universal History with a Cosmopolitan Purpose," Kant put forward the view that nature is concerned with seeing that man should work his way onwards to make himself worthy of life and well-being. He added: "What remains disconcerting about all this is firstly, that the earlier generations seem to perform their laborious tasks only for the sake of the later ones, so as to prepare for them a further stage from which they can raise still higher the structure intended by nature; and secondly, that only the later generations will in fact have the good fortune to inhabit the building on which a whole series of their forefathers...had worked without being able to share in the happiness they were preparing." See Reiss [24] (p.44).

variables. For example, health, education, and biodiversity should not fall below certain levels, while emissions of pollutants should not exceed a certain level. These environmental constraints, when they are effective, induce some "costs" in term of welfare growth. In the climate change debate, a ceiling of green house gases concentration would impose restriction on the current growth pattern as emissions would have to be curtailed. This is the cost of providing future generations the right to live in a more or less tolerable climate. When defining such an environmental constraint, current generations trade off this cost and the level of the environmental objective they agree to sustain for future generations.

It is well recognized that if floors are too high and ceilings are too low, the set of possible actions will be empty. Assuming that the set of feasible actions is not empty, there is still the question of trade-offs between floors and ceilings. Martinet [19] described the trade-offs between several sustainability objectives (i.e., quantities that should be sustained), without considering welfare or growth concerns.[2] The thresholds are interpreted as *minimal rights* to be guaranteed to all generations. Alvarez-Cuadrado and Long [1] examined the implication of a floor on consumption on the growth path of a society that optimally chooses its floor.

In this paper, we propose a criterion for ranking social alternatives, based on an indicator called "Rights and Welfare Indicator" (RWI for short). This indicator combines a welfare index (based on the conventional utilitarian objective of maximizing the integral of the discounted stream of utility derived from the consumption of goods and services) with an index of rights, such as the right to satisfy basic needs or the right to have access to natural capital and to biodiversity. The index of rights is an aggregate measure of various thresholds representing "sustainability" in a broad sense. As in Martinet [19], our index of rights is an index of the threshold levels, not of the extent to which society exceeds the various thresholds. This index is non-decreasing in each threshold level. It is likely that increasing any threshold will reduce the welfare index. In this sense, there is a tension between rights and welfare.

We explore the implications of this approach by the means of examining the RWI. We illustrate these implications on the path of resource use. We put forward the view that society may not seek to maximize "welfare" (in a standard sense offered by welfare eco-

[2] In particular, Martinet does not address the question of how to rank growth paths that satisfy the optimally selected combination of minimal rights, when several paths are sustainable in his sense.

nomics), but instead may be also concerned with rights, and may maximize an index that consists of two sub-indices, one related to welfare and the other related to rights. Society thus makes trade-offs between welfare and rights. Maximizing the value of the Rights and Welfare Indicator (which is not a measure of social welfare) is a way to represent these trade-offs.

Our paper is related to the paper by Alvarez-Cuadrado and Long [1], which proposed a "Mixed Bentham Rawls Criterion" that seeks a trade off between a utilitarian indicator and the consumption level of the worse-off generation. If the latter is interpreted as a *minimal consumption right*, one may argue that the model of [1] is a special case of our model, where there are many minimal rights. However, [1] did not refer to the above interpretation.

Following Martinet [19], we assume that, for each right, it is possible to construct an indicator function showing at each point of time how a society is faring in terms of meeting that right. A threshold for an indicator is the numerical level below which the indicator is not allowed to fall. An indicator is a function of a set of state variables and control variables. For example, a possible indicator for adult literacy could be the percentage of adults who can write and read at a certain level of proficiency; a possible indicator for biodiversity could be the number of species not in serious danger of immediate disappearance. Since the maintenance of an indicator above a threshold level typically requires the use of resources, it is plausible to argue that for any given level of resource endowment, there is a well-defined "threshold possibility frontier," which is the upper boundary of a "threshold possibility set." While Martinet focused on the choice of thresholds on the threshold possibility frontier, we allow for the possibility that a society may choose to be inside the frontier, because the cost of being on the frontier, measured in terms of forgone consumption of some goods and services, may outweigh the value of guaranteeing a high level of the rights represented by the thresholds.

The threshold possibility frontier delimits a set of feasible thresholds within which a vector of "optimal thresholds" would be chosen. The optimal threshold vector precisely balances the "costs" of thresholds in terms of welfarist consequences (e.g., lower consumption for some generations), and the "moral worth" of thresholds. While the trade-offs are captured by a scalar measure, the latter should not be interpreted as a measure of "generalized welfare."

We show that, depending on the preferences and the relative weight accorded to minimal rights, the optimal development path may either be a constrained utilitarian path, or switch to a development path fully characterized by the minimal rights guaranteed to all generations ("*right-based sustainable development*"). When the minimal rights constraints are effective, social discount rate is different from the classical utilitarian formulation.

The remaining of the paper is organized as follows. The motivation of our approach is detailed in Section 2. We present therein the tension between rights and welfare, as well as a brief history of sustainability criteria that puts our criterion in perspective. Section 3 presents the implication of the studied criterion in a finite time framework. The results are illustrated in a model of exhaustible resource allocation. Section 4 presents the implication of the studied criterion in an infinite time framework. The results are illustrated in the Dasgupta-Heal-Solow model of nonrenewable resource depletion and capital accumulation. Section 5 gathers the implications of our results and our conclusions.

2 Motivation

2.1 Rights versus Welfare

The tension between rights and welfarist considerations has been a subject of debate among philosophers, thinkers, and economists. The Rawlsian theory of justice places rights above welfare.[3] In fact, Rawls's first principle of justice is that everyone should have equal rights: "each person is to have an equal right in the most extensive scheme of equal basic liberties compatible with a similar scheme of liberties for others." His second principle of justice, the difference principle, insists that social and economic inequalities are acceptable only if they are arranged so that they are "both (a) to the greatest expected benefit of the least advantaged and (b) attached to offices and positions open to all under conditions of fair equality of opportunity." In particular, difference in income is acceptable only if it improves the life prospects of the least advantaged. Rawls acknowledges that

[3]Rawls's conception of justice has its foundation in the theory of social contract advanced by Locke, Rousseau, and Kant. The initial position conceived by Rawls is a hypothetical situation in which the contracting parties are individuals hidden behind the veil of ignorance: none of them knows his place in society, his natural talents, intelligence, strength, and the like. In other words, the principles of justice are agreed to in an initial situation that is fair.

economic growth is necessary, because without adequate material resources a society cannot develop institutions that guarantees equal liberties to all. He points out that the difference principle must be modified to allow for economic growth, as a unmodified difference principle would lead to "no savings at all." The need for adequate savings is a major concern for Rawls, because, "to establish effective, just institutions within which the basic liberties can be realized"society must have a sufficient material base. Generations must "carry their fair share of the burden of realizing and preserving a just society."Rawls sketches a theory of "just saving" to modify the difference principle.[4] Wealth creation is necessary for the effective defense of rights and liberties.

Another influential philosopher who stresses the preponderance of rights is Nozick [21]. He emphasized the importance of property rights, from a somewhat different perspective. Nozick's work has inspired alternative articulations of libertarian rights with a game-theoretic flavor.[5] In our paper, we abstract from game-theoretic considerations.

Different from the right-based approach to development is the welfarist approach. This latter is based on intertemporal welfare functions (i.e., criteria) describing the intertemporal performance of the economy.

2.2 A short history of sustainability criteria

The criterion studied in this paper is formally a generalization of the criterion proposed by Alvarez-Cuadrado and Long [1], taking into account several rights and sustainability indicators, as in Martinet [19]. To explain the emergence of such a criterion, and the way it gathers rights and welfare in a unified framework, we present a short history of sustainability criteria.

To describe the criteria, we consider a continuous time framework, and assume that the economy is composed of infinitely many generations, to focus on intergenerational equity. We thus make the simplifying assumption that each generation can be assimilated to a representative agent, and do not address intragenerational equity. Let x be a vector of n state variables, and c a vector of m control variables. Denote the instantaneous utility function by $U(x(t), c(t), t)$. The transition equations are $\dot{x}_k(t) = g_k(x(t), c(t), t)$,

[4] See [16].

[5] See [12, 30, 11, 10, 13, 14, 22, 32], among others. For an overview, see Suzumura [31]. These papers acknowledge the fundamental contribution of Sen [25, 26, 27].

for $k = 1, 2, ..., n$. Given the values of the state variables, the control variables at time t must belong to a technologically feasibility set $A(x(t), t)$ which is characterized by a set of s inequality constraints:

$$h_j(x(t), c(t), t) \geq 0,\ j = 1, 2, ..., s. \tag{1}$$

For state and decisions, a continuous path is denoted by $x(\cdot)$ or $c(\cdot)$. Given an initial state $x(0) = x_0$, a given continuous path of admissible decisions $c^a(\cdot)$ generates a single continuous path of economic states $x^a(\cdot)$.

The traditional criterion for evaluating intertemporal paths is the discounted utility criterion

$$W^{DU}(x_0, c^a(\cdot)) = \int_0^\infty e^{-\delta t}\ U(x^a(t), c^a(t), t)\ dt\ , \tag{2}$$

where $\delta > 0$ is the constant discount rate. According to this criterion, an economic path starting from initial state x_0 and generated by the decision path $c^1(\cdot)$ is strictly preferred to an alternative path starting from the same initial state and generated by decisions $c^2(\cdot)$ if and only if $W^{DU}(x_0, c^1(\cdot)) > W^{DU}(x_0, c^2(\cdot))$. A decrease in the utility level of a generation (no matter how disadvantaged this generation already is and how large is the considered sacrifice) can be justified by a sufficient increase in the utility level of some other generations. This criterion is strongly inequitable, and has been shown to display "dictatorship of the present," a term coined by Chichilnisky [6]. For example, in the case of the Dasgupta-Heal-Solow model, the optimal consumption path under discounted utilitarianism decreases toward zero in the long run [7]. Defining a criterion that accounts for intergenerational equity, and in particular for the long run, has been a challenge of sustainability economics.

An alternative criterion, which is anonymous, is the maximin criterion [4, 28]:

$$W^{Mm}(x_0, c^a(\cdot)) = \inf_t \left(U(x^a(t), c^a(t), t)\right)\ . \tag{3}$$

According to this criterion, $c^1(\cdot)$ is strictly preferred to $c^2(\cdot)$ if and only if $\inf_t \left(U(x^1(t), c^1(t), t)\right) > \inf_t \left(U(x^2(t), c^2(t), t)\right)$. Many economists (e.g., [6]) have pointed out that the maximin criterion is insensitive to the utilities of generations that are not the poorest. According to the maximin welfare function (3), an increase in the utility of any generation that is not the least advantaged does not raise social welfare W^{Mm}.[6] Moreover, if it is possible to smooth utility over time, the maximin principle leads to no growth, no matter how small is the initial maximal sustainable utility. There is no concern for growth, which may be an issue if capital accumulation is needed to develop and sustain just institutions.

By applying the idea of the golden rule of economic growth to the sustainability issue, one can define a development path that reaches and sustains the highest possible development level. The resulting criterion is termed Green Golden Rule [3], and considers only the very long run:

$$W^{GGR}(x_0, c^a(\cdot)) = \lim_{t\to\infty} U(x^a(t), c^a(t), t) \ . \tag{4}$$

It has been qualified as a dictatorship of the future by Chichilnisky [6]. Two development paths generated by decisions $c^1(\cdot)$ and $c^2(\cdot)$ are compared only with respect to their limiting behavior.

The welfare function proposed by Chichilnisky [6] is a weighted sum of two terms. The first term being the usual discounted stream of utilities. The second term is defined in a way that its value depends only on the limiting behavior of the utility sequence.[7] Formally,

$$W^{CHI}(x_0, c(\cdot)) = (1-\theta)\int_0^\infty \lambda(t)U(x(t), c(t), t)dt + \theta \lim_{t\to\infty} U(x(t), c(t), t) \ , \tag{5}$$

where $0 < \theta < 1$, $0 < \lambda(t) \le 1$, $\int_0^\infty \lambda(t) < \infty$. This criterion is neither a dictatorship of the present nor a dictatorship of the future. It, however, has some limitations (such as the non-existence of a solution for some simple problems).

Alvarez-Cuadrado and Long [1] proposed to modify the Chichilnisky criterion replacing the second term with the minimal level of utility of the trajectory over time. The resulting social welfare function is denoted by W^{MBR}, where the superscript MBR stands for

[6]The maximin criterion has been strengthened to eliminate some maximin paths that are Pareto dominated by other paths that have the same minimum level of utility. See [2, 17].

[7]The original formulation of the criterion is in discrete time and ranks infinite sequences of utility. To be consistent within the present paper, we give the continuous time equivalent.

"Mixed Bentham-Rawls":

$$W^{MBR}(x_0, c(\cdot)) = (1-\theta)\int_0^\infty e^{-\delta t} U\left(x(t), c(t), t\right) dt + \theta \inf_t \left(U(x(t), c(t), t)\right) \ , \qquad (6)$$

where $0 < \theta < 1$. This social welfare function is a weighted average of the standard sum of discounted utilities and a Rawlsian part, which places special emphasis on the utility of the least advantaged generation. The positive weight $(1-\theta)$ on the discounted utilitarian part implies non-dictatorship of the future, just as it does for Chichilnisky's welfare function.[8] The positive weight θ on the Rawlsian part ensures non-dictatorship of the present.[9]

All the criteria presented above weigh the welfare of the various generations differently. This has strong implications in terms of the discounting. More specifically, the discount rate to be used to evaluate project investment with long run impacts is strongly influenced by the criterion chosen.

The mixed Bentham-Rawls criterion is in sharp contrast to the standard utilitarian tradition (see, e.g., any graduate macro-economic textbook) which would treat a family line as an infinitely-lived individual. Such a textbook position could result in requiring great sacrifices of early generations who are typically poor. In contrast, the MBR criterion avoids imposing very high rates of savings at the earlier stages of accumulation. As regards sustainability concerns, the maximization of the MBR criterion determines endogenously a minimal utility level to be sustained forever. The criterion introduces the idea that all generations (and in particular future generations) have some "right" to enjoy a minimal utility, and that their welfare cannot be sacrificed too much for other generations (in particular present generations). This approach is, however, still utilitarian, and focused on intergenerational equity and the "weight" given to each generation.

In a quite different perspective, Martinet [19] examined a criterion defining several sustainability thresholds. A finite number (I) of sustainability issues are represented by indicators $\mathcal{I}_i(x(t), c(t))$ and thresholds μ_i. These thresholds are interpreted as *minimal*

[8]It also implies that social welfare is increasing in U_t, ensuring that the strong Pareto property is satisfied. The utility of the least advantaged is thus not the only thing that matters. One may say that this rules out "dictatorship of the least advantaged."

[9]The MBR criterion, just as Chilchilnisky's criterion, is based on cardinal utility. It can be generalized, so that the discount rate can be variable, as in Chilchilnisky's case.

rights to be guaranteed to all generations. There are no intergenerational trade-offs. All generations have the same minimal rights with respect to the sustainability issues considered. The objective is not to weigh the different generations in an intertemporal welfare function, but to define minimal rights representing sustainability. The achievable thresholds are traded off to determine what is guaranteed to all generations, with the following sustainability criterion:

$$\begin{aligned} \max_{\mu_i,\ i=1,\ldots,I} \quad & \mathcal{P}(\mu_1, \ldots, \mu_I) \ , \\ \text{s.t} \quad & \mathcal{I}_i(x(t), c(t)) \geq \mu_i \ , \qquad i = 1, \ldots, I \ ; \forall t \in \mathbb{R}^+ \ , \\ & \dot{x} = g(x(t), c(t)) \ , \\ & x(0) = x_0 \ . \end{aligned} \tag{7}$$

Martinet's approach focuses on a set of minimal rights, without considering welfare.[10] The approach proposed in our paper consists in modifying the criterion (6) by changing the minimal utility over time by an index of sustainability thresholds as in problem (7). The proposed criterion combines a welfare index with an index based on minimal rights.[11] The levels of the minimal rights are defined endogenously, and come at a "cost" in term of present-value welfare.

In the following sections, we examine the implications of this criterion in a finite time framework (Section 3) and in an infinite time framework (Section 4).

3 Finite horizon

Consider first the case of a finite horizon T. The initial stocks $x_k(0)$, $k = 1, 2, ..., n$, are given. The terminal stocks are free, subject to $x_k(T) \geq 0$.

Define $\mathcal{F}(x_0; \mu_1, ..., \mu_I)$ as the set of all the economic paths $(x(\cdot), c(\cdot))$ starting from initial state x_0 and satisfying all the constraints defined by the indicators and the thresholds

[10] Note that the criterion in [19] is ordinal.

[11] Combining a cardinal welfare index with an index based on minimal rights requires defining a cardinal measure of rights.

at all times, i.e.,

$$\mathcal{F}(x_0;\mu_1,\ldots,\mu_I)=\left\{(x(\cdot),c(\cdot))\;\middle|\;\begin{array}{l} x_k(0)=x_{k0}\ , x_k(T)\geq 0\ ,\\ \dot{x}_k=g_k(x(t),c(t),t)\ ,\quad k=1,\ldots,n\ ;\forall t\in[0,T]\\ h_j(x(t),c(t),t)\geq 0\ ,\quad j=1,\ldots,s\ ;\forall t\in[0,T]\\ \mathcal{I}_i(x(t),c(t))\geq\mu_i\ ,\quad i=1,\ldots,I\ ;\forall t\in[0,T]\end{array}\right\} \tag{8}$$

Clearly, given the initial stock x_0, the set $\mathcal{F}(x_0;\mu_1,...,\mu_I)$ may be empty if the thresholds μ_i are too high. It is sensible to consider only thresholds that are consistent with the economic endowment x_0. For this purpose, let us define the *set of feasible thresholds,*

$$\mathcal{M}(x_0)=\{(\mu_1,\ldots,\mu_I)|\ \mathcal{F}(x_0;\mu_1,\ldots,\mu_I)\neq\emptyset\}\ .$$

Assume that the upper boundary of the set $\mathcal{M}(x_0)$ can be represented by the equality $\phi(\mu_1,\ldots,\mu_I;x_0)=0$, and that points below this frontier yield $\phi(\mu_1,\ldots,\mu_I;x_0)>0$, where ϕ is a differentiable function.[12]

Assume a constant rate of discount $\delta\geq 0$. A feasible time path $(x(t),c(t))$ starting from state x_0 yields a welfare indicator

$$W(x(\cdot),c(\cdot))\equiv\int_0^T e^{-\delta t}U\left(x(t),c(t),t\right)dt$$

where $U(.,.,.)$ is the instantaneous utility function.

We suppose that society places values on the minimal rights guaranteed at all times, i.e., on thresholds μ_i, $i=1,\ldots,I$. This valuation is represented by a function $\mathcal{P}(\mu_1,\ldots,\mu_I)$ that is increasing in each argument μ_i.

Our Rights and Welfare Indicator (RWI) is defined by $J=\theta\mathcal{P}(\mu_1,\ldots,\mu_I)+(1-\theta)W(x(\cdot),c(\cdot))$, where $0<\theta<1$ is the relative weight given to "rights." The parameter θ is taken as given (it can be interpreted as the political weight of the "non-welfarist" proponents). We propose that society maximizes the Rights and Welfare Indicator J:

[12] We assume here the existence of a function ϕ that is differentiable, to represent the boundary of the set of feasible minimal rights. Our examples will clarify that point.

$$J(x_0, c(\cdot), \mu) = \theta \mathcal{P}(\mu_1, \ldots, \mu_I) \; + \; (1-\theta) \int_0^T e^{-\delta t} U\,(x(t), c(t), t)\, dt \; . \tag{9}$$

To maximize the RWI given the vector of initial stocks $x_0 \equiv (x_{10}, x_{20}, ..., x_{n0})$, the planner chooses the thresholds levels, i.e., the numbers $(\mu_1, \ldots, \mu_I) \in \mathcal{M}(x_0)$, and the time path $c(\cdot)$ to maximize the above objective function, over all feasible paths given by the set $\mathcal{F}(x_0; \mu_1, ..., \mu_I)$ defined by eq. (8).

The objective function highlights the potential *tension between rights and welfare.* For example, maximizing welfare would call for present-bias consumption smoothing, with the utility path tilted toward the present if δ is high enough; but such present-bias consumption smoothing may not be desirable if the emphasis on the right of future generations to have a minimal consumption is very strong.[13]

3.1 The necessary conditions

Since $(\mu_1, \ldots, \mu_I)$ are constants to be chosen optimally, the optimization problem (9) is an optimal control problem with $(\mu_1, \ldots, \mu_I)$ treated as control parameters. The necessary conditions for such problems can be derived from Hestenes' Theorem.[14] They are as follows.

Necessary conditions for optimization of the Rights and Welfare Indicator *Let $\pi(t)$ denote the vector of co-state variables, $\lambda(t)$ the vector of multipliers associated with the technological inequality constraints $h_j(x(t), c(t), t) \geq 0, \quad j = 1, \ldots, s,$, and $\omega(t)$ the vector of multipliers associated with the right-based constraints*

$$\mathcal{I}_i(x(t), c(t)) - \mu_i \geq 0, \quad i = 1, \ldots, I \; . \tag{10}$$

[13]Note that our framework could be extended to consider non-negotiable basic rights. It could be done either by assuming that there is a set of (exogenous) numbers $(z_1, \ldots, z_I) \in \mathcal{M}(x_0)$ such that the planner's choice of the thresholds $(\mu_1, \ldots, \mu_I) \in \mathcal{M}(x_0)$ must also satisfy the non-negotiable basic right constraints $\mu_i \geq z_i$, or by taking into account such strong sustainability constraints in the definition of the preference function $\mathcal{P}(\mu_1, \ldots, \mu_I)$, as suggested in [19] (p.190).

[14]See [18, Theorem 7.11.1] or [33] for an exposition of Hestenes' Theorem which deals with optimal control problems involving control parameters and various constraints.

The Hamiltonian for this problem is

$$H(t, x(t), c(t), \pi(t)) \equiv (1-\theta)e^{-\delta t}U(x(t), c(t), t) + \pi(t)g(x(t), c(t), t) \ ,$$

and the Lagrangian is

$$L(t, x(t), c(t), \pi(t), \lambda(t), \omega(t), \mu) = H + \lambda(t)h(x(t), c(t), t) + \omega(t)\left[\mathcal{I}(x(t), c(t)) - \mu\right] \ .$$

An optimal path must satisfy the following conditions:[15]

(i) The control variables maximize the Hamiltonian subject to the inequality constraints (1) and (10)., i.e., $dL/dc = 0$.

(ii) $\dot{\pi}_k = -\partial L/\partial x_k, \quad k = 1, \ldots, n.$

(iii) $\dot{x}_k = \partial L/\partial \pi_k, \quad k = 1, \ldots, n.$

(iv) The transversality conditions for the optimal choice of the control parameters $\mu_i, \quad i = 1, \ldots, I,$ *are*

$$\theta \frac{\partial \mathcal{P}}{\partial \mu_i} + \int_0^T \frac{\partial L}{\partial \mu_i} dt + \gamma \frac{\partial \phi}{\partial \mu_i} = 0 \ , \tag{11}$$

with

$$\gamma \geq 0 \ , \ \phi(\mu_1, \ldots, \mu_I; x_0) \geq 0, \text{ and } \gamma\phi(.) = 0 \ ,$$

and the transversality conditions for the optimal choice of the final stocks are $x_k(T) \geq 0$, $\pi_k(T) \geq 0$, $\pi_k(T)x_k(T) = 0$, *for* $k = 1, \ldots, n.$[16]

(v) The Hamiltonian and the Lagrangian are continuous functions of time, and, along the optimal path,[17]

$$\frac{d}{dt}H(t, x(t), c(t), \pi(t)) = \frac{d}{dt}L(t, x(t), c(t), \pi(t), \lambda(t), \omega(t), \mu) = \frac{\partial L}{\partial t} \ .$$

[15] We here consider the first order, necessary conditions only, for the sake of simplicity. The sufficiency conditions (concavity conditions) can be derived as in [18]. We also assume that constraint qualifications are satisfied (see [33]).

[16] We use the terms "transversality conditions" as Hestenes for expressions involving the choice of control parameters.

[17] I.e., the value of the total derivative of L (along the optimal path) equals the value of the partial derivative $\partial L/\partial t$, evaluated at the optimal vectors of controls, states, and multipliers. This is a necessary condition along intervals of time where H is differentiable with respect to time. See [18] or [33].

3.2 An example: Exhaustible resource exploitation with joint product technology

Consider an economy with an initial stock of an exhaustible resource $S_0 > 0$. Let $r(t) \geq 0$ denote the rate of extraction. Then $\dot{S}(t) = -r(t)$. We require that $S(T) \geq 0$.

The economy uses $r(t)$ as an input to produce two consumption goods, denoted by $c_1(t)$ and $c_2(t)$ as "joint products" under the production function[18]

$$\frac{1}{2}c_1^2 + \frac{1}{2}c_2^2 = r^2$$

(e.g., if it chooses to set $c_1 = 0$, then $c_2 = r\sqrt{2}$). Consumption levels c_1 and c_2 must be non-negative.

Allowing free disposal, we can write our production function $\frac{1}{2}c_1^2 + \frac{1}{2}c_2^2 = r^2$ more generally as

$$h_1(r, c_1, c_2) = r^2 - \frac{1}{2}\left(c_1^2 + c_2^2\right) \geq 0 \ .$$

Note that, given T and S_0, the *maximum feasible* constant consumption of each good is $\bar{c}_i = \frac{S_0}{T}\sqrt{2}$, $i = 1, 2$. Let there be two sustainability indicators, and associated thresholds

$$\mathcal{I}_1(c_1, c_2, S) \equiv c_1 \geq \mu_1 \ ,$$

$$\mathcal{I}_2(c_1, c_2, S) \equiv c_2 \geq \mu_2 \ .$$

For example, if $S_0 = 20$ and $T = 10$, then one can achieve (by setting $c_2(t) = 0$ for all t) the highest feasible constant consumption level for good 1, $\bar{c}_1 = 2\sqrt{2}$, by choosing $r = \frac{S_0}{T} = 2$ for all $t \in [0, 1]$. If we set the threshold levels at $\mu_1 = 2\sqrt{2}$ and $\mu_2 = 0$ (e.g., if sustaining the level of indicator $\mathcal{I}_2(c_1, c_2, S)$ is not considered to be very important), then a feasible path exists. But if threshold levels are too high, e.g., $\mu_1 = 2\sqrt{2} = \mu_2$, then the set of feasible paths $\mathcal{F}(S_0 = 20; \mu_1 = 2\sqrt{2}, \mu_2 = 2\sqrt{2})$ is empty. If we set $\mu_1 = \mu_2 = 2$, then a feasible path exists, by choosing $r(t) = 2$ for all $t \in [0, 10]$ and setting $c_1(t) = 2 = c_2(t)$ for all $t \in [0, 10]$.

[18]The joint product technology assumed in this example may well reflect the choice of output levels made by a professor for a given input level r. For instance, c_1 is the number of research papers per year, and c_2 is the number of graduate students per year, while r is "effort" level. Alternatively, for any given quantity of oil (r), various levels of heating (c_1) and transport (c_2) can be achieved.

The conditions of this problem imply a feasibility set $\phi(\mu_1, \mu_2; S_0) = \left(\frac{S_0}{T}\right)^2 - \frac{1}{2}\mu_1^2 - \frac{1}{2}\mu_2^2 \geq 0$. The upper boundary of the set of feasible thresholds $\mathcal{M}(S_0)$ is represented by the equation $\phi(\mu_1, \mu_2; S_0) = 0$, where $\mu_i \geq 0, \ i = 1, 2$.

The RWI to be maximized is

$$J = \theta \mathcal{P}(\mu_1, \mu_2) \ + \ (1-\theta) \int_0^T e^{-\delta t} U(c_1(t), c_2(t)) dt \ .$$

Suppose that

$$\mathcal{P}(\mu_1, \mu_2) = \ln \mu_1 + \ln \mu_2 \ ,$$

and

$$U(c_1, c_2) = 2c_1^{1/2} + 2c_2^{1/2} \ .$$

3.2.1 Characterization of the optimal solution

The characterization of the optimal solution allows us to state the following proposition:

Proposition 1 (Constant consumption in the presence of positive discounting) *If the relative weight of Rights, $\theta/(1-\theta)$, exceeds a certain critical value, then the social optimum calls for constant consumption despite positive discounting. This critical value is an increasing function of the discount rate δ.*

Proof of Proposition 1: Let us derive all the necessary conditions. Write the (present value) Hamiltonian

$$H(t, S(t), c_1(t), c_2(t), r(t), \pi(t)) \equiv (1-\theta) e^{-\delta t} \left[2c_1(t)^{1/2} + 2c_2(t)^{1/2}\right] - \pi(t) r(t) \ ,$$

and the (present value) Lagrangian

$$L = H + \lambda(t) \left[r(t)^2 - \frac{1}{2} \left(\ c_1(t)^2 + c_2(t)^2 \right) \right] + \omega_1(t) \left[c_1(t) - \mu_1 \right] + \omega_2(t) \left[c_2(t) - \mu_2 \right] \ .$$

The FOCs are

$$\frac{\partial L}{\partial r} = -\pi(t) + 2\lambda(t) r(t) = 0 \ , \tag{12}$$

$$\frac{\partial L}{\partial c_1} = (1-\theta) e^{-\delta t} \frac{1}{\sqrt{c_1(t)}} - \lambda(t) c_1(t) + \omega_1(t) = 0 \ , \tag{13}$$

$$\frac{\partial L}{\partial c_2} = (1-\theta)e^{-\delta t}\frac{1}{\sqrt{c_2(t)}} - \lambda(t)c_2(t) + \omega_2(t) = 0 \ ,$$

$$\omega_i(t)\left[c_i(t) - \mu_i\right] = 0, \omega_i(t) \geq 0, c_i(t) - \mu_i \geq 0 \ , , \ i = 1,2$$

$$\lambda(t)\left[r(t)^2 - \frac{1}{2}\left(\ c_1(t)^2 + c_2(t)^2\right)\right] = 0, r(t)^2 - \frac{1}{2}\left(\ c_1(t)^2 + c_2(t)^2\right) \geq 0, \lambda(t) \geq 0 \ ,$$

$$\dot{\pi}(t) = -\frac{\partial L}{\partial S} = 0 \ , \ \text{implying } \pi(t) = \overline{\pi} \text{ (a constant).}$$

The transversality conditions are

$$S(T) \geq 0, \pi(T) \geq 0, \pi(T)S(T) = 0 \ ,$$

and $\theta\frac{\partial \mathcal{P}}{\partial \mu_i} + \int_0^T \frac{\partial L}{\partial \mu_i}dt + \gamma\frac{\partial \phi}{\partial \mu_i} = 0$, with $\gamma \geq 0$, $\phi(\mu_1, \mu_2; S_0) \geq 0$, and $\gamma\phi(.) = 0$, implying

$$\frac{\theta}{\mu_i} - \gamma\mu_i = \int_0^T \omega_i(t)dt \ , \ i = 1,2 \ . \tag{14}$$

Using the fact that

$$2\lambda(t)r(t) = \pi(t) = \overline{\pi} \iff \lambda(t) = \frac{\overline{\pi}}{2r(t)} \ ,$$

we obtain from (13)

$$\omega_i(t) = \overline{\pi}\frac{c_i(t)}{2r(t)} - (1-\theta)e^{-\delta t}\frac{1}{\sqrt{c_i(t)}} \text{ for all } t \in [0,T] \ , i = 1,2 \ . \tag{15}$$

Substitute this into (14) we get, for $i = 1,2$,

$$\frac{\theta}{\mu_i} - \gamma\mu_i = \int_0^T \overline{\pi}\frac{c_i(t)}{2r(t)}dt - \int_0^T (1-\theta)e^{-\delta t}\frac{1}{\sqrt{c_i(t)}}dt \ . \tag{16}$$

Using symmetry, $c_1 = c_2$ and $\mu_1 = \mu_2$. Let us show that if $\theta/(1-\theta)$ is large enough, the solution will be constant consumption, with $r(t) = c_1(t) = c_2(t) = c^*$ (a constant) where

$$c^* = \mu^* = \left(\frac{S_0}{T}\right) \ . \tag{17}$$

We must show that the solution (17) satisfies all the necessary conditions. Substitute for $c_i(t) = c^*$ in equations (15) and (16):

$$\omega(t) = \frac{\overline{\pi}}{2} - (1-\theta)e^{-\delta t}\frac{1}{\sqrt{\mu^*}} \ , \tag{18}$$

$$\frac{\theta}{\mu^*} - \gamma\mu^* = \frac{\overline{\pi}}{2}T - \frac{(1-\theta)}{\sqrt{\mu^*}}\left[\frac{1-e^{-\delta T}}{\delta}\right] \ .$$

Since a necessary condition is $\omega(t) \geq 0$, we must check that (18) is non-negative for all $t \in [0, T]$. Hence $\overline{\pi}$ must be chosen such that

$$\frac{\overline{\pi}}{2} - (1-\theta)\frac{1}{\sqrt{\mu^*}} \geq 0 \ .$$

Choose $\overline{\pi}$ such that this condition holds with equality, i.e., $\frac{\overline{\pi}}{2} = (1-\theta)\frac{1}{\sqrt{\mu^*}}$. Then,

$$\begin{aligned}\frac{\theta}{\mu^*} - \gamma\mu^* = (1-\theta)\frac{1}{\sqrt{\mu^*}}T - \frac{1-\theta}{\sqrt{\mu^*}}\left[\frac{1-e^{-\delta T}}{\delta}\right] &= (1-\theta)\frac{1}{\sqrt{\mu^*}}\left[\frac{\delta T - (1-e^{-\delta T})}{\delta}\right] \\ &\geq 0 \qquad \text{for all } \delta \geq 0 \text{ and } T > 0 \ ,\end{aligned}$$

with strict inequality if $\delta > 0$ and $\theta < 1$. Then

$$\gamma^* = \frac{1}{\mu^*}\left[\frac{\theta}{\mu^*} - (1-\theta)\frac{1}{\sqrt{\mu^*}}\left(\frac{\delta T - (1-e^{-\delta T})}{\delta}\right)\right] \ .$$

The necessary condition (11) requires that $\gamma^* \geq 0$. This condition is satisfied if and only if

$$\frac{\theta}{\mu^*} \geq (1-\theta)\frac{1}{\sqrt{\mu^*}}\left(\frac{\delta T - (1-e^{-\delta T})}{\delta}\right) \ . \tag{19}$$

Clearly, in the special case where $\theta = 1$, condition (19) is satisfied. But even if $\theta < 1$, this condition is also satisfied as long as

$$\frac{\theta}{1-\theta} \geq \sqrt{\mu^*}\left[\frac{\delta T - (1-e^{-\delta T})}{\delta}\right] \equiv \sqrt{\left(\frac{S_0}{T}\right)}\left[\frac{\delta T - (1-e^{-\delta T})}{\delta}\right] \ . \tag{20}$$

NOTE: $\left[\frac{\delta T-(1-e^{-\delta T})}{\delta}\right]>0$ for all $\delta>0$ and $T>0$. Using L'Hopital's rule, we can show that $\frac{\delta T-(1-e^{-\delta T})}{\delta}=0$ if $\delta\to 0$. Note also that $\left[\frac{\delta T-(1-e^{-\delta T})}{\delta}\right]$ is increasing in δ. Suppose that for given θ , S_0 and T, there exists a value of δ, say $\overline{\delta}>0$ such that eq. (20) holds with equality. Then if we increase δ beyond the threshold $\overline{\delta}$, the constant consumption path (17) will cease to be an optimal solution. ■

3.2.2 Implications for discounting

In this subsection, we discuss the policy implications of maximizing the RWI, by offering some economic interpretations of the optimality conditions. From equations (12) and (13), we obtain the social optimal condition

$$-\delta+\frac{d\ln U'_{c_1}}{dt}=\frac{d\ln}{dt}\left[\overline{\pi}\left(\frac{c_1}{2r}\right)-\omega_1\right] .$$

On the other hand, if individuals are price-takers in a perfectly competitive capital market, their intertemporal consumption smoothing (without regards for the thresholds) implies that

$$-\frac{d\ln U'_{c_1}}{dt}=\left(\rho_1(t)-\delta\right) ,$$

where $\rho_1(t)$ is the rate of interest facing the consumers (in terms of the consumption good c_1). It follows that if the planner's allocation is to be achieved by a decentralized mechanism, the implied rate of interest facing the consumers must satisfy

$$\rho_1(t)=-\frac{d\ln}{dt}\left[\overline{\pi}\left(\frac{c_1}{2r}\right)-\omega_1\right] \tag{21}$$

where

$$\frac{c_1}{2r}=-\frac{h'_c}{h'_r}$$

is the marginal cost of consumption c_1 in terms of the resource input.

In particular, if the solution involves constant consumption (symmetric for both goods), then

$$\overline{\pi}\left(\frac{c_1}{2r}\right)=2\lambda c ,$$

and thus

$$\overline{\pi}\left(\frac{c_1}{2r}\right) - \omega_1 = 2\lambda c - \omega_1 = (1-\theta)e^{-\delta t}\frac{1}{\sqrt{c_1}} ,$$

i.e., using (21), the rate of interest offered to consumers are

$$\rho_1 = \delta$$

while the rate of interest offered to producers is zero. This wedge between producer's interest rate and consumer's interest rate implies an interest subsidy to consumers, to counter their natural inclination of tilting consumption toward the present.

As private individuals, consumers tend to discount future consumption too much, violating the constraint on consumption rights of future generations. An interest subsidy counters this incentive by encouraging them to save.

4 Infinite horizon

Suppose the time horizon is infinite and the rate of discount δ is a positive constant. Then the social planner chooses μ and $c(\cdot)$ to maximize the objective function:

$$J(x_0, c(\cdot), \mu) = \theta\mathcal{P}(\mu_1, \ldots, \mu_I) + \int_0^\infty (1-\theta)U(x,c,t)e^{-\delta t}dt .$$

That is, the planner maximizes

$$\int_0^\infty \left\{\theta\mathcal{P}(\mu_1, \ldots, \mu_I)\delta + (1-\theta)U(x,c,t)\right\} e^{-\delta t}dt . \tag{22}$$

4.1 Necessary conditions

Let $\psi(t) = e^{\delta t}\pi(t)$, $\Delta(t) = e^{\delta t}\lambda(t)$ and $w(t) = e^{\delta t}\omega(t)$. The current-value Hamiltonian of this infinite horizon problem is

$$H^c = \theta\mathcal{P}(\mu_1, \ldots, \mu_I)\delta + (1-\theta)U(x,c,t) + \psi g(x,c,t) ,$$

and the current-value Lagrangian is

$$L^c = H^c + \Delta h(x,c,t) + w\left[\mathcal{I}(x,c,t) - \mu\right].$$

The first-order conditions of the optimization problem are as follows.

$$\begin{aligned}
\frac{\partial L^c}{\partial c} = (1-\theta)U'_c + \psi g'_c + \Delta h'_c + w\mathcal{I}'_c = 0\ , \\
\Delta \geq 0, \quad h(x,c,t) \geq 0, \quad \Delta h(x,c,t) = 0\ , \\
w \geq 0, \quad \mathcal{I}(x,c,t) - \mu \geq 0, \quad w\left[\mathcal{I}(x,c,t) - \mu\right] = 0\ , \\
\dot{\psi} = \delta\psi - \frac{\partial L^c}{\partial x}\ , \\
\dot{x} = \frac{\partial L^c}{\partial \psi}\ .
\end{aligned} \tag{23}$$

The optimality conditions with respect to the control parameters μ_i, for $i = 1, \ldots, I$, are

$$\int_0^\infty e^{-\delta t}\frac{\partial L^c}{\partial \mu_i}dt + \gamma\frac{\partial \phi}{\partial \mu_i} = 0\ , \tag{24}$$

with $\gamma \geq 0$, $\phi(\mu_1, \ldots, \mu_I; x_0) \geq 0$, and $\gamma\phi(.) = 0$.

Finally, the transversality conditions with respect to the stocks are

$$\lim_{t\to\infty} e^{-\delta t}\psi(t) \geq 0, \text{ and } \lim_{t\to\infty} e^{-\delta t}\psi(t)x(t) = 0.$$

4.2 An example: The production-consumption economy with a nonrenewable resource

Consider the Dasgupta-Heal-Solow model of nonrenewable resource extraction and capital accumulation [7, 8, 28]. Capital stock is denoted by $K(t)$, resource stock by $S(t)$, resource extraction by $r(t)$ and consumption by $c(t)$. We assume a Cobb-Douglas production function, i.e., $F(K,r) = K^\alpha r^\beta$. The dynamics of this economy are as follows:

$$\dot{K}(t) = K(t)^\alpha r(t)^\beta - c(t), \tag{25}$$

$$\dot{S}(t) = -r(t). \tag{26}$$

We consider the following sustainability indicators of consumption and resource stock,[19]

$$\begin{aligned}\mathcal{I}_1(c,r,S,K) &\equiv c\ ,\\ \mathcal{I}_2(c,r,S,K) &\equiv S\ ,\end{aligned}$$

as well as the following rights/sustainability constraints (as in [19, 20]):

$$c(t) \geq \mu_c\ , \tag{27}$$

$$S(t) \geq \mu_S\ . \tag{28}$$

These constraints state that every generation has the right to a minimal consumption at level μ_c, and the right to a minimal preserved stock μ_S.

The set of achievable minimal consumption and preserved resource stock (μ_c, μ_S) is characterized by the following relationship (see [19, 20]):

$$\phi(\mu_c,\mu_S,K_0,S_0) \equiv (1-\beta)\big((S_0-\mu_S)(\alpha-\beta)\big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}} - \mu_c \geq 0\ . \tag{29}$$

The upper boundary of this set satisfies $\phi(\mu_c,\mu_S,K_0,S_0) = 0$. It can be represented by the following "threshold possibility frontier":

$$\mu_c = (1-\beta)\big((S_0-\mu_S)(\alpha-\beta)\big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}}\ . \tag{30}$$

This curve has a negative slope and is concave, for all $\mu_S < S_0$:

$$\frac{\partial \mu_c}{\partial \mu_S} = -\beta\,(S_0-\mu_S)^{\frac{-1}{1-\beta}}\,\big(\alpha-\beta\big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}} < 0\ ,$$

$$\frac{\partial^2 \mu_c}{(\partial \mu_S)^2} = -\frac{\beta}{1-\beta}\,(S_0-\mu_S)^{\frac{-2+\beta}{1-\beta}}\,\big(\alpha-\beta\big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}} < 0.$$

[19]Several authors have used the production-consumption economy to address the climate change issue (e.g., [9, 29]). The nonrenewable resource is related to fossil energy. Stabilizing green house gas (GHG) concentrations requires limiting the cumulative emissions over time. The in-ground resource stock is used as a proxy for non-emitted GHG. A limit on cumulative emissions can be represented by a constraint on resource extraction: a part of the stock has to be preserved.

4.2.1 The RWI criterion for $U(c(t), S(t)) \equiv U(c(t))$ and $\mathcal{P}(\mu_c, \mu_S) \equiv \eta_c\mu_c + \eta_S\mu_S$

Assume that $\mathcal{P}(\mu_c, \mu_S) \equiv \eta_c\mu_c + \eta_S\mu_S$ (where η_c and η_S are non-negative parameters), and that instantaneous utility is derived only from consumption, i.e., $U(c(t))$.

Consider the objective function

$$(1-\theta)\int_0^\infty e^{-\delta t}U(c(t))dt + \theta\left[\eta_c\mu_c + \eta_s\mu_S\right] \ , \tag{31}$$

subject to

$$\dot{K}(t) = K(t)^\alpha r(t)^\beta - c(t) \, , \, K(0) = K_0, \, K(t) \geq 0 \ , \tag{32}$$

$$\dot{S}(t) = -r(t), \ S(0) = S_0, \ \lim_{t\to\infty} S(t) \geq \mu_S \ , \tag{33}$$

$$c(t) - \mu_c \geq 0 \ , \tag{34}$$

$$S(t) - \mu_S \geq 0 \ , \tag{35}$$

and

$$\phi(\mu_c, \mu_S, S_0, K_0) \equiv (1-\beta)\big((S_0-\mu_S)(\alpha-\beta)\big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}} - \mu_c \geq 0 \ .$$

The objective is then equivalent to maximize the expression

$$\int_0^\infty \left\{\delta\theta(\eta_c\mu_c + \eta_S\mu_S) + (1-\theta)U(c(t))\right\} e^{-\delta t} dt \ .$$

The current value Hamiltonian is

$$H^c = (1-\theta)U(c(t)) + \theta\delta(\eta_c\mu_c + \eta_S\mu_S) + \psi_K\left[K(t)^\alpha r(t)^\beta - c(t)\right] - \psi_S r(t) \ .$$

The Lagragian is

$$L^c = H^c + w_c(c-\mu_c) + w_S(S-\mu_S) \ .$$

The necessary conditions of this problem are

$$
\begin{aligned}
\frac{\partial L^c}{\partial c} = 0 \quad &\Leftrightarrow \quad (1-\theta)U'_c - \psi_K + w_c = 0 \ , && (36)\\
\frac{\partial L^c}{\partial r} = 0 \quad &\Leftrightarrow \quad \psi_S = \psi_K F'_r \ , && (37)\\
\dot{\psi}_k = \delta\psi_K - \frac{\partial L^c}{\partial K} \quad &\Leftrightarrow \quad \frac{\dot{\psi}_K}{\psi_K} = \delta - F'_K \ , && (38)\\
\dot{\psi}_S = \delta\psi_S - \frac{\partial L^c}{\partial S} \quad &\Leftrightarrow \quad \dot{\psi}_S = \delta\psi_S - \omega_S \ , && (39)\\
\int_0^\infty e^{-\delta t}\frac{\partial L^c}{\partial \mu_c}dt + \gamma\frac{\partial \phi}{\partial \mu_c} = 0 \quad &\Leftrightarrow \quad \theta\eta_c - \int_0^\infty e^{-\delta t} w_c dt - \gamma = 0 \ , && (40)\\
\int_0^\infty e^{-\delta t}\frac{\partial L^c}{\partial \mu_S}dt + \gamma\frac{\partial \phi}{\partial \mu_S} = 0 \quad &\Leftrightarrow \quad \theta\eta_S - \int_0^\infty e^{-\delta t} w_S dt \ldots \\
&- \quad \gamma\beta(\alpha-\beta)^{\frac{\beta}{1-\beta}}(S_0-\mu_S)^{\frac{\beta}{1-\beta}-1}K_0^{\frac{\alpha-\beta}{1-\beta}} = 0 \ , && (41)
\end{aligned}
$$

with $\gamma \geq 0$, $\phi(\mu_c, \mu_S; S_0, K_0) \geq 0$ and $\gamma\phi(.) = 0$, as well as conditions (32), (33), (34), (35), and

$$w_c \geq 0,\ w_c(c-\mu_c) = 0 \ ,$$

$$w_S \geq 0,\ w_S(S-\mu_S) = 0 \ .\text{(I have replaced } \omega_S \text{ by } w_S\text{)}$$

Remark 1 (Logarithmic valuation of rights) *If we had specified $P(\mu_c, \mu_S) \equiv \eta_c \ln \mu_c + \eta_S \ln \mu_S$, then the terms $\theta\eta_c$ and $\theta\eta_S$ in the last two equations, (40) and (41) would have to be replaced by $\theta\eta_c/\mu_c$ and $\theta\eta_S/\mu_S$. All other equations would remain unchanged.*

4.2.2 Characterization of the optimal solution

Interestingly, condition (41) implies the following proposition.

Proposition 2 *The optimal solution of the RWI maximization must be one of three types:*

1. *If $\mu_S^\star = S_0$ the solution is straightforward as the whole resource stock is preserved, and there is no consumption. The solution $(\mu_c^\star, \mu_S^\star)$ is on the threshold possibility frontier, and corresponds to the corner solution $(\mu_c^\star, \mu_S^\star) = (0, S_0)$. Consumption is positive for some finite time, since the model allows the eating up of the capital stock.*

2. *If $\mu_c^\star > 0$, then the solution $(\mu_c^\star, \mu_S^\star)$ is on the threshold possibility frontier, and consumption is constant if $\mu_S^\star > 0$.*

3. *If $\mu_c^\star = 0$, then the solution $(\mu_c^\star, \mu_S^\star)$ may be in the interior of the set of feasible thresholds.*

Proof of part 2 of Proposition 2

If $\mu_c^\star > 0$, then $c(t) > 0$ for all t, which implies positive production, and thus positive extraction for all t.[20] The resource stock will thus be declining at any time $t \in [0, \infty)$, and the constraint $S \geq \mu_S^\star$ will never be binding. It is as if an amount $\mu_S^\star$ is set aside, and the remaining amount, $S_0 - \mu_S^\star > 0$ is extracted, with exhaustion occurring only in the asymptotic sense. The associate shadow value w_S is then nil at all times. Assuming that $\eta_S > 0$, condition (41) can be satisfied only if $\gamma > 0$, requiring that $\phi(.) = 0$, which means that the solution $(\mu_c^\star, \mu_S^\star)$ is on the *threshold possibility frontier*. Let us show that $c(t) = \mu_c^\star$ for all t (if $\mu_S^\star > 0$ and $\mu_c^\star > 0$). Suppose that $c(t) > \mu_c^\star + \varepsilon$ over some time interval, where ε is some strictly positive number. Then by re-arranging investment and consumption, it is feasible to ensure that $c(t) > \mu_c^\star + \frac{1}{n}\varepsilon$ for some number $n > 0$ for all t. But we have shown that the solution $(\mu_c^\star, \mu_S^\star)$ is on the threshold possibility frontier. Given $\mu_S^\star > 0$, the inequality $c(t) > \mu_c^\star + \frac{1}{n}\varepsilon$ would contradict the result that $(\mu_c^\star, \mu_S^\star)$ is on the threshold possibility frontier. It follows that $c(t) = \mu_c^\star$ for all t (if $\mu_S^\star > 0$ and $\mu_c^\star > 0$).[21] ■

Proposition 2 implies that an interior solution of our problem is possible only if there is no strictly positive consumption guaranteed, i.e., $\mu_c^\star = 0$. A part $\mu_S^\star < S_0$ of the stock may then be preserved. In all other cases, the solution is on the *threshold possibility frontier*, and consumption is always constant at the level $\mu_c^\star$, corresponding to the maximin consumption under the preservation constraint $\mu_S^\star$. We devote the next two subsections to study, respectively (i) the binding solution and (ii) the conditions under which $\mu_c^\star = 0$. As we don't know *a priori* which case corresponds to the optimal solution of our general problem, we differentiate the "optimal" candidates of each case by using a symbol different

[20] In the DHS model, to have consumption always bounded below by a floor $\underline{c} > 0$, the resource input must be positive for all $t \in [0, \infty)$.

[21] Note that, if $\eta_S = 0$, then $\gamma = 0$ and therefore the solution can be in the interior of the feasibility set: it would be if $\delta > 0$, i.e., when $\delta > 0$ and $\theta \in (0, 1)$, the maximin solution would not be optimal.

from the optimality mark * used in the Proposition (respectively, * for the binding solution, and $\hat{}$ for the interior solution).

Remark 2 *In the above proof, we showed that if* $(\mu_c^\star, \mu_S^\star) > (0,0)$ *is on the threshold possibility frontier, then* $c(t) = \mu_c^\star$ *for all* t*. Then eq. (36) becomes*

$$(1-\theta)U_c'(\mu_c^\star) - \psi_K(t) + w_c(t) = 0 \ ,$$

i.e.,

$$\psi_K(t) - w_c(t) = \ \textit{constant} = (1-\theta)U_c'(\mu_c^\star) \equiv \chi$$

This means that

$$w_c(t) = \psi_K(t) - \chi$$

where χ *is some positive constant. Thus, using condition (38),*

$$w_c(t) = \psi_K(0)\exp\left[\int_0^t (\delta - F_K'(\tau))d\tau\right] - \chi$$

Substituting this into eq. (40)

$$\theta\eta_c + \frac{\chi}{\delta} - \int_0^\infty \psi_K(0)\exp\left[-\int_0^t F_K'(\tau))d\tau\right]dt - \gamma = 0$$

and using (41), and the definition of χ

$$\theta\eta_c + \frac{(1-\theta)U_c'(\mu_c^\star)}{\delta} - \int_0^\infty \psi_K(0)\exp\left[-\int_0^t F_K'(\tau)d\tau\right]dt = \frac{\theta\eta_S}{\beta(\alpha-\beta)^{\frac{\beta}{1-\beta}}(S_0-\mu_S^\star)^{\frac{\beta}{1-\beta}-1}K_0^{\frac{\alpha-\beta}{1-\beta}}}$$

where

$$(1-\beta)\big((S_0-\mu_S^\star)(\alpha-\beta)\big)^{\frac{\beta}{1-\beta}}K_0^{\frac{\alpha-\beta}{1-\beta}} = \mu_c^\star$$

These two equations determine $(\mu_c^\star, \mu_S^\star)$ *if* $\psi_K(0)\exp\left[-\int_0^t (F_K'(\tau))d\tau\right]$ *is known.*

Remark 3 *In the case of logarithmic valuation of rights (see Remark 1), it is clear that the optimal* $\mu_c^\star$ *and* $\mu_S^\star$ *are both strictly positive. Hence only part 2 of Proposition 2 applies in this case.*

4.2.3 The "binding" solution

Assume that $\eta_S > 0$. We can characterize the optimal thresholds (μ_c^*, μ_S^*) when they are chosen on the *threshold possibility frontier*, $\phi(\mu_c, \mu_S, K_0, S_0) = 0$.

In this case, it follows from part 2 of Proposition 2 that the solution corresponds to the maximin consumption under a resource preservation constraint [4, 19, 20, 28]. The consumption is constant, at a level

$$\begin{aligned} c^+(K_0, S_0, \mu_S^*) &= (1-\beta)\big((S(t)-\mu_S^*)(\alpha-\beta)\big)^{\frac{\beta}{1-\beta}} K(t)^{\frac{\alpha-\beta}{1-\beta}} \\ &= (1-\beta)\big((S_0-\mu_S^*)(\alpha-\beta)\big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}} \\ &= \mu_c^* \,. \end{aligned} \tag{42}$$

It yields a net present value $NPV = \frac{1}{\delta}U(\mu_c^*)$ and the constraints yield a sustainability value $\mathcal{P}(\mu_c^*, \mu_S^*)$, so that the maximized RWI level is $J^* = (1-\theta)\frac{1}{\delta}U(\mu_c^*) + \theta\mathcal{P}(\mu_c^*, \mu_S^*)$.

We know μ_S^* as a function of μ_c^* when these parameters are on the boundary of the feasibility set from the expression $\phi = 0$. We can define the function $\mu_S^* = \bar{\mu}_S(\mu_c^*)$ from eq. (42).

From the expression of J, and the condition on the optimal choice of the parameters on the boundary, we can derive the solution. It satisfies the following condition:[22]

$$\frac{dJ}{d\mu_c} = 0 \quad \Leftrightarrow$$

$$\underbrace{(1-\theta)\frac{1}{\delta}U'(\mu_c^*)}_{\substack{\text{Net present value gain} \\ \text{from increasing the} \\ \text{constant consumption level}}} + \underbrace{\theta\mathcal{P}'_{\mu_c}(\mu_c^*, \bar{\mu}_S(\mu_c^*))}_{\substack{\text{Gain in terms of} \\ \text{guaranteed consumption}}} = \underbrace{\theta\bar{\mu}'_S(\mu_c^*)\mathcal{P}'_{\mu_S}(\mu_c^*, \bar{\mu}_S(\mu_c^*))}_{\text{Loss in terms of preserved stock}} \tag{43}$$

It is shown in the appendix that this feasible solution may satisfy the optimal conditions of the original optimization problem. We shall discuss in subsection 4.2.5 the conditions on the preference parameters for this solution to be optimal.

[22] Providing an explicit expression of the optimal thresholds is possible from this condition given a specific utility function.

4.2.4 No guaranteed consumption

We now turn to the case 3 of Proposition 2, and consider the "optimal" solution when $\hat{\mu}_c = 0$. This is the only case that allows the optimal choice $(\hat{\mu}_c, \hat{\mu}_S)$ to be not on the *threshold possibility frontier*, $\phi(\mu_c, \mu_S, K_0, S_0) = 0$.

The optimal trajectory of this case is described in the appendix. The nature of the solution depends on the value of the marginal utility of consumption when consumption is nil (i.e., $U'(0)$). We distinguish two cases: The finite marginal utility case and the infinite marginal utility case. In the finite marginal utility case, it is shown that consumption is positive over a finite time interval, after which the economy stays at a stationary state with no consumption, no capital stock, and the preservation of a resource stock $\hat{\mu}_S$. In the infinite marginal utility case, the consumption is positive at all times, and a part of the stock $(S_0 - \hat{\mu}_S)$ is depleted asymptotically.

Whatever the case, it is possible to define some welfare value function $V(S_0 - \mu_S, K_0)$, depending on the preservation constraint threshold μ_S, which satisfies:

$$
\begin{aligned}
V(S_0 - \mu_S, K_0) \quad &= \quad \max_{c(\cdot), r(\cdot)} \int_0^\infty U(c(t)) e^{-\delta t} dt \ , \\
s.t. \qquad &\quad \dot{K}(t) = K(t)^\alpha r(t)^\beta - c(t) \ , \\
&\quad \dot{S}(t) = -r(t) \ , \\
&\quad K(0) = K_0 \ , \\
&\quad S(0) = S_0 - \mu_S \ .
\end{aligned}
\tag{44}
$$

The optimal conservation threshold $\hat{\mu}_S$ must solve

$$
\max_{\mu_S} J(\mu_S) \equiv (1-\theta) V(S_0 - \mu_S, K_0) + \theta \eta_S \mu_S \ . \tag{45}
$$

Assuming that the previous value function can be characterized, the optimal conservation level $\hat{\mu}_S$ satisfies

$$
\frac{dJ}{d\mu_S} = 0 \quad \Leftrightarrow
$$

$$
\underbrace{-(1-\theta)V_S'(S_0 - \hat{\mu}_S, K_0)}_{\substack{\text{Net present value loss}\\ \text{from increasing the}\\ \text{preservation constraint}}} \quad + \quad \underbrace{\theta\eta_S}_{\substack{\text{Gain}\\ \text{in terms of}\\ \text{preserved stock}}} \quad = 0 \tag{46}
$$

which is equivalent to

$$
\frac{\partial}{\partial \mu_S}(V(S_0 - \hat{\mu}_S, K_0)) = -\frac{\theta}{(1-\theta)}\eta_S \ . \tag{47}
$$

We cannot characterize further the expression of $\hat{\mu}_S$ without knowing the expression of the value function.[23] We can say, however, that there is a unique solution, as the value function is monotonic increasing and concave in the states if utility is strictly increasing and concave consumption.[24]

Moreover, corner solution are not excludable. On the one hand, if $V_S'(S_0, K_0) \geq \frac{\theta}{(1-\theta)}\eta_S$, it is optimal to preserve none of the resource stock, i.e., $\hat{\mu}_S = 0$. This case corresponds to the unconstrained utilitarian solution. On the other hand, if $V_S'(0, K_0) \leq \frac{\theta}{(1-\theta)}\eta_S$, it is optimal to preserve all the initial resource stock, i.e., $\hat{\mu}_S = S_0$. This case corresponds to case 1 in the lemma.

4.2.5 Parameter conditions

We can then ask what the values $\theta, \eta_c, \eta_S, \delta$ are, such that $(\mu_c^\star, \mu_S^\star)$ is on the frontier, i.e., $(1-\beta)\big((S_0 - \mu_S^\star)(\alpha-\beta)\big)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}} - \mu_c^\star = 0$. In particular, we can ask if there is a range of values of θ for which an interior solution occurs.

We have some $\hat{\mu}_S(\theta, \delta, \eta_S)$ on the one hand, and a $\mu_c^*(\theta, \delta, \eta_c, \eta_S)$ and the associated $\mu_S^*(\theta, \delta, \eta_c, \eta_S) = \bar{\mu}_S(\mu_c^*)$ on the other hand.

[23] It is usually not possible to have a close-form solution to problem (44). We provide an example of characterization of this value function in the appendix.

[24] For a proof, see [15].

Each candidate provides a RWI as follows:

$$\hat{J} = (1-\theta)V(S_0 - \hat{\mu}_S, K_0) + \theta\eta_S\hat{\mu}_S$$

and

$$J^* = (1-\theta)\frac{1}{\delta}U(\mu_c^*) + \theta(\eta_c\mu_c^* + \eta_S\mu_S^*)$$

We can say that $(\mu_c^\star, \mu_S^\star) = (0, \hat{\mu}_S)$ if $\hat{J} > J^*$. It is hard to go further without the expressions of the various candidates (and the value function). Our results, however, suggest that there are some parameters value for which the solution is a discounted utility path with conservation of a part of the resource, and other values for which the solution is driven by the minimal consumption and resource preservation rights.

Fig. 1 illustrates these two cases. Note that for $\mathcal{P}(\mu_c, \mu_S) \equiv \eta_c\mu_c + \eta_S\mu_S$, the iso-value RWI curves correspond to planes in the space of welfare index and rights (with relative slopes depending on $(1-\theta)$, $\theta\eta_c$, and $\theta\eta_S$).

4.2.6 Implications for discounting

Let us offer some economic interpretations of the necessary conditions. In the absence of minimal-rights constraints, we would have the following familiar efficiency conditions. First, the Hotelling rule states that the resource price rises at an exponential rate equal to the interest rate facing producers (the marginal product of capital), i.e., $\frac{1}{F_r'}\frac{d(F_r')}{dt} = F_K'$. Second, the Keynes-Ramsey rule states that the rate of growth of consumption is equal to the product of the elasticity of intertemporal substitution $\sigma \equiv \frac{-U_c'}{cU_c''}$ and the difference between the interest rate facing consumers, $\rho(t)$, and the utility discount rate δ. In a competitive economy without externalities and policy intervention, the consumption rate of interest $\rho(t)$ is equal to the marginal productivity of capital. The Keynes-Ramsey rule reads $\frac{\dot{c}}{c} = \sigma(F_K' - \delta) = \sigma(\rho(t) - \delta)$. This rule can also be expressed as follows,

$$\frac{\dot{U}_c'}{U_c'} \equiv -\frac{1}{\sigma}\left(\frac{\dot{c}}{c}\right) = \delta - \rho(t)\ ,$$

and tells us that the consumption increases over time (i.e., the rate of change of marginal utility is negative and marginal utility decreases) if the consumption discount rate (the

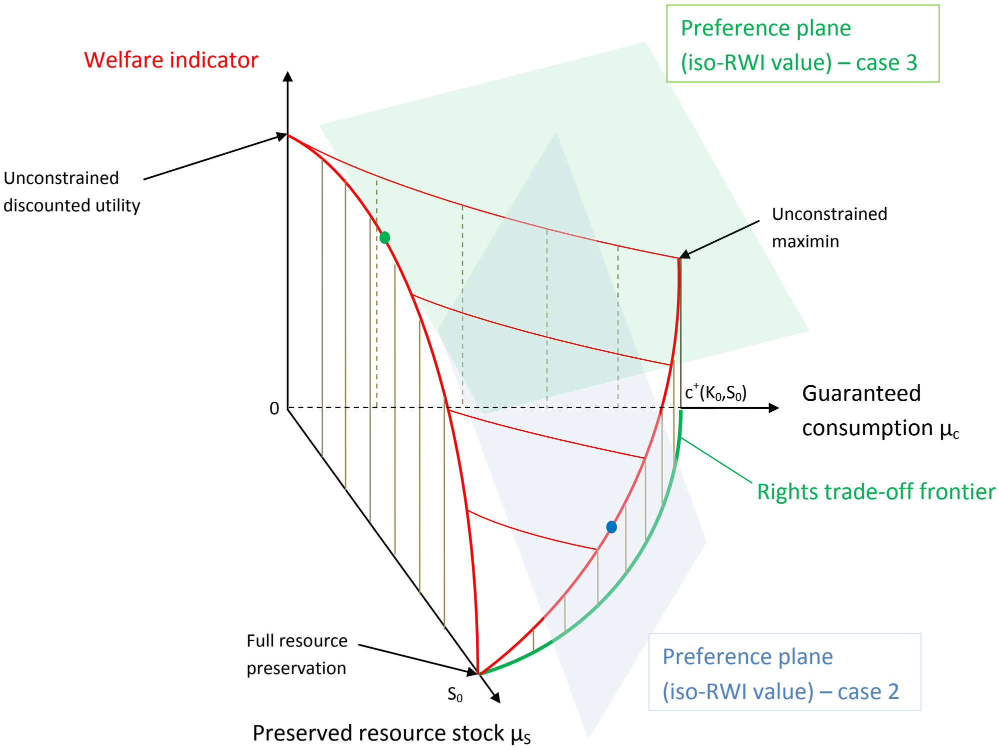

Figure 1: Trade-offs between welfare and rights in the DHS model.

interest rate) is larger than the impatience represented by the utility discount rate. Alternatively, expressing the consumption discount rate as a function of the utility discount rate, the growth rate and the elasticity of intertemporal substitution, i.e.,

$$\rho(t) = \delta + \frac{1}{\sigma}\frac{\dot{c}}{c}\ ,$$

one gets the usual expression of the discount rate to apply to investment project. It is equal to the sum of pure preference for the present plus the wealth effect.

When the minimal right constraints are binding, these conditions are modified.

If the resource preservation constraint is binding, the dual variable w_S is positive, and we have a “modified Hotelling Rule”:

$$\frac{1}{F_r'}\frac{d}{dt}(F_r') = \frac{\dot{\psi}_S}{\psi_S} - \frac{\dot{\psi}_K}{\psi_K} = \left(\delta - \frac{w_S}{\psi_S}\right) - (\delta - F_K') = F_K' - \frac{w_S}{\psi_S} \leq F_K' \, .$$

That is, if some resource stock is to be kept in the ground after a certain time T, when extraction stops, it must be the case that the resource owners find that, after time T, the price at which they can sell the resource as an input, namely $F_r'(r(\tau), K(\tau))$ for $\tau > T$, does not rise fast enough to compensate for the loss of interest income. (Presumably, at T they can sell the remaining resource stock to the government to keep it in the ground for ever, at the price $F_r'(r(T), K(T))$).

When the guaranteed consumption constraint is not binding, $w_c = 0$ and we get the usual Keynes-Ramsey rule. When the constraint is binding, the wealth effect is modified.

If the minimal consumption constraint is binding, the dual variable w_c is positive, and one has a “modified Keynes-Ramsey Rule”:

$$\begin{aligned}
-\frac{1}{\sigma}\left(\frac{\dot{c}}{c}\right) &= \frac{1}{\psi_K - w_c}\frac{d}{dt}\left(\psi_K - w_c\right) \\
&= \left(\frac{\psi_K}{\psi_K - w_c}\right)\left(\frac{\dot{\psi}_K}{\psi_K}\right) - \left(\frac{w_c}{\psi_K - w_c}\right)\left(\frac{\dot{w}_c}{w_c}\right) \\
&= \left(\frac{\psi_K}{\psi_K - w_c}\right)[\delta - F_K'] - \left(\frac{w_c}{\psi_K - w_c}\right)\left(\frac{\dot{w}_c}{w_c}\right) \\
&\equiv \delta - \rho^c(t)
\end{aligned}$$

Thus, the implicit interest rate that consumers should use to discount consumption is $\rho^c(t)$ defined by

$$\rho^c(t) = \delta - \left(\frac{\psi_K}{\psi_K - w_c}\right)[\delta - F_K'] + \left(\frac{w_c}{\psi_K - w_c}\right)\left(\frac{\dot{w}_c}{w_c}\right)$$

On the other hand, let $\rho^I(t)$ be the interest rate used to discount the future returns on investment. It is equal to the marginal product of capital. Then

$$\rho^I(t) = F_K' = \delta + \frac{1}{\sigma}\frac{\dot{c}}{c}\left(\frac{\psi_K - w_c}{\psi_K}\right) - \frac{\dot{w}_c}{\psi_K} \neq \rho^c(t) = \delta + \frac{1}{\sigma}\frac{\dot{c}}{c}$$

This wedge between producer's interest rate and consumer's interest rate implies tax or subsidy on savings, to ensure minimal consumption rights.

For example, if it is socially optimal to have constant consumption for ever, then the implicit interest facing private households should be $\rho^c(t) = \delta$, while the interest rate facing producers is F'_K (which is greater than δ earlier in the program, when the capital stock is low, and less than δ later in the program, when the capital stock is high, along the constant consumption path of the DHS model).

5 Concluding Remarks

The present paper introduces a criterion that accounts for Rights and Welfare in ranking social alternatives of development paths. The criterion is a weighted sum of a welfare index and an index of minimal rights guaranteed to all generations. Such a criterion could represent the choice of a democracy where the RWI reflects the preference of voters. The miminal rights are chosen by the voters (who are homogeneous in our model). At the same time, we assume that when individuals make their own private decisions (e.g., how much to consume, how much to bequeath to their children) they are not individually guided by their concern for rights. These latter are implemented by the elected government.

Our examples illustrate the possibility that, at some point, minimal rights are so important that the path of feasible trajectories is reduced to a single path, and the willingness to satisfy these minimal rights intertemporally drives the development path (right-based sustainable development).

We have also shown that the necessary conditions yield implications about discount rate to be applied on investment projects. In particular, satisfying minimal consumption may imply some wedge between consumers and producers interest rates, possibly implemented by tax or subsidy on savings.

Acknowledgments: We thank the participants of the CIREQ Natural Resource Economics workshop (Montreal, January 2012) and IFO lunch time seminar (Munich, January, 2012). We thank Bob Cairns, Hassan Benchekroun, Hans-Werner Sinn, and Karen Pittel for comments. All errors are our owns.

A Appendix

A.1 Binding case: Optimality of the feasible path

Let us show that the described feasible solution is optimal. Consider a given $\mu_c^* > 0$. Because $\mu_S^* = \bar{\mu}_S(\mu_c^*)$, it is not feasible to have a phase $[0, T]$ where $c(t) > \mu_c^*$ for all $t \in [0, T]$.

In other words, given $\mu_c^* > 0$ and $\mu_S^* = \bar{\mu}_S(\mu_c^*)$, consider the following problem:

$$\max(1-\theta)\int_0^{\infty} e^{-\delta t} U(c) dt \tag{48}$$

s.t.

$$\dot{K} = K(t)^{\alpha} r(t)^{\beta} - c(t)\,,\; K(0) = K_0,\; K(t) \geq 0$$

$$\dot{S} = -r(t),\;\; S(0) = S_0,\;\; \lim_{t\to\infty} S(t) \geq \bar{\mu}_S(\mu_c^*)$$

$$c - \mu_c^* \geq 0$$

By construction, we know this problem has a feasible solution (as described above) where $c(t) = \mu_c^*$ for all t, and the value of this feasible program is $\frac{1}{\delta}U(\mu_c^*)$. But does this feasible solution satisfy the necessary conditions for problem (48)? Let us check (we use the superscript M to distinguish this problem from problem (31)). The necessary conditions for problem (48) are derived below.The current value Hamiltonian is

$$H^M = (1-\theta)U(c(t)) + \psi_k^M \left[K(t)^{\alpha} r(t)^{\beta} - c(t)\right] - \psi_S^M r(t)$$

The Lagragian is

$$L^M = H^M + w_c^M (c - \mu_c^*)\;.$$

The necessary conditions of this problem are

$$\frac{\partial L^M}{\partial c} = 0 \Leftrightarrow (1-\theta)U_c' - \psi_K^M + w_c^M = 0$$

$$\frac{\partial L^M}{\partial r} = 0 \Leftrightarrow \psi_S^M = \psi_K^M F_r'$$

$$\dot{\psi}_K^M = \delta\psi_K^M - \frac{\partial L^M}{\partial K} \Leftrightarrow \frac{\dot{\psi}_K^M}{\psi_K^M} = \delta - F'_K$$

$$\dot{\psi}_S = \delta\psi_S - \frac{\partial L^c}{\partial S} \Leftrightarrow \dot{\psi}_S = \delta\psi_S$$

and also (32), $\dot{S} = -r(t), \quad S(0) = S_0$, $\lim_{t\to\infty} S(t) \geq \bar{\mu}_S(\mu_c^*), c - \mu_c^* \geq 0$, and

$$\omega_c^M \geq 0, w_c^M[c - \mu_c^*] = 0$$

Setting $c(t) = \mu_c^*$ for all t, we have

$$(1-\theta)U'_c(\mu_c^*) - \psi_K^M + w_c^M = 0 \ ,$$

which implies

$$w_c^M(t) = \psi_K^M(t) - \text{constant}$$

which is OK as long as $\psi_K^M(t)$ is never smaller than that constant.

A.2 Interior solution: Characterization of the optimal paths

A.2.1 Finite marginal utility case

If the solution is not on the boundary of the feasibility set $\phi(.) \geq 0$, one has $\gamma = 0$. In this case, condition (40) leads to $\int_0^\infty e^{-\delta t}\omega_c dt = \theta\eta_c$. Since η_c is positive by assumption, it follows from the above equality that w_c must be positive over some time interval. Some generation will experience the minimal level of consumption, i.e., $\hat{c}(t) = 0$ for some t.

Using a similar argument, we conclude that the constraint $S(t) \geq \mu_S$ must be binding, i.e. there exists some T such that for all $t \geq T$ the stock remains at μ_S forever. Combining these two requirements, this means that there is some time T from which the economy stops using the resource, producing and consuming. We term this part of the path "phase 2." During phase 1 (positive consumption), the dynamics are driven by exactly the same conditions as the Dasgupta-Heal solution (discounted utility).

To solve the problem, we proceed backward. We first characterize phase 2, to obtain the terminal conditions of phase 1. We then solve the phase 1 problem, treating the time T as a parameter to optimize.

Phase 2: Starting from some time T, assume a stationary state at stock $S(T) = \mu_S$ without extraction and consumption.

We have a stationary state with both sustainability constraints binding. The associated dual variables are positive. The necessary conditions (40) and (41) are not very helpful: for any T, there are many (non-stationary) paths w_c and w_S satisfying these conditions, with no other implications. We deduce from condition (36) that $\psi_k = (1-\theta)U'(0) + w_c(T)$ but we cannot determine $w_c(T)$.

Since capital has no use after T, we expect that all the capital stock is gradually eaten up before T is reached, i.e., $\lim_{t\to T} K(t) = 0$. After time T, the marginal products F'_r and F'_K are not defined (the marginal products depend on the factor ratio r/K which is not defined after T). Making use of information before time T, we have the following system of three differential equations: the Keynes-Ramsey Rule,

$$\frac{d\ln U'(c)}{dt} = \delta - F'_K ,$$

the Hotelling Rule,

$$F'_K = \frac{d\ln F'_r}{dt} = \alpha\frac{\dot{K}}{K} + (\beta - 1)\frac{\dot{r}}{r} ,$$

and the transition equation

$$\dot{K} = F - c .$$

Together with the three boundary conditions

$$\int_0^T r(t)dt = S_0 - \mu_S ,$$

$$K(0) = K_0 ,$$

$$K(T) = 0 ,$$

we can determine (in principle) the time path of (c^*, K^*, r^*) for given T and μ_S. (NOTE: we do not impose that $c(T) = 0$).

This solution path (c^*, K^*, r^*) yields the welfare indicator

$$W(\mu_S, T; K_0, S_0) = \int_0^T e^{-\delta t} U(c^*(t)) dt \ .$$

Given that $\mu_c = 0$, the Rights and Welfare indicator is

$$J = (1-\theta) W(T; K_0, S_0 - \mu_S) + \eta_S \mu_S$$

Maximizing J with respect to μ_S and T determines the optimal length of Phase 1 and the optimal μ_S^*. The FOC are

$$(1-\theta)\frac{\partial W(T; K_0, S_0 - \mu_S)}{\partial \mu_S} + \eta_S = 0 \iff (1-\theta)\psi_S(0) = \eta_S$$

$$(1-\theta)\frac{\partial W(T; K_0, S_0 - \mu_S)}{\partial T} = 0 \iff (1-\theta) H^c(T) = 0$$

This condition(and the continuity of $H(t)$) implies that

$$\lim_{t \to T} \left[U(c(t)) + \psi_k(t) \left[F(t) - c(t) \right] - \psi_S(t) r(t) \right] = 0$$

If $U(c) = 0$ at $c = 0$, this condition is consistent with

$$\lim_{t \to T} c(T) = 0.$$

A.2.2 Infinite marginal utility case

In the case where $U'(0) = \infty$, the phase 2 described in the previous case would not exist. In this case there is some $\mu_S > 0$ that is set aside from the beginning. To determine μ_S we can proceed as follows.

Consider the discounted utility maximization a la Dasgupta and Heal, and the associated value function for an initial stock of resource $S_0 - \mu_S$:

$$V(S_0 - \mu_S, K_0) \equiv \max_{c,r} \int_0^\infty e^{-\delta t} U(c(t)) dt \ , \tag{49}$$

s.t.

$$\dot{K} = K(t)^{\alpha} r(t)^{\beta} - c(t)\,,\ K(0) = K_0,\ K(t) \geq 0\,,$$

$$\dot{S} = -r(t),\ \ S(0) = S_0,\ \ \lim_{t\to\infty} S(t) = \mu_S\ .$$

This function can in principle be calculated (though not in closed form).[25]

Value function for a special case of the Dasgupta-Heal model Suppose the social planner wants to treat all individuals symmetrically and seeks to maximize the life-time utility of the representative individual

$$\max \int_0^{\infty} e^{-\delta t} \left[\frac{1}{1-\gamma} c^{1-\gamma} \right] dt \tag{50}$$

subject to

$$\dot{K} = \left(K^{\alpha} r^{\beta} - c \right),\ K(0) = K_0$$

$$\dot{S} = -r,\ S(0) = S_0$$

$$K \geq 0$$

$$S \geq 0$$

Let $V(K,S)$ be the value function of the social planner's problem. The Hamilton-Jacobi-Bellman equation is, after substitution,

$$\delta V(K,S) = \max \left[\frac{1}{1-\gamma} c^{1-\gamma} + V_k \left(K^{\alpha} r^{\beta} - c_i \right) - V_S r \right]$$

To get an analytical solution, we make the following assumptions on parameter values:

Assumption A1: $\gamma = \alpha$

Let us conjecture that, for $K > 0$ and $S > 0$, the value function takes the form

$$V(K,S) = AK^{1-\alpha} + BS^{\beta}$$

[25] For some special cases of problem (49), it is possible to obtain a closed form solution for the value function. In this case, using the expression of the value function, it is possible to solve explicitely problem (45).

Then we have

$$\delta V(K,S) = \max\left[\frac{c^{1-\alpha}}{1-\alpha} + A(1-\alpha)K^{-\alpha}\left(K^{\alpha}r^{\beta} - c\right) - B(\beta)S^{\beta-1}r\right] \tag{51}$$

The FOCs with respect to c and R are

$$c^{-\alpha} = A(1-\alpha)K^{-\alpha}$$

$$A(1-\alpha)r^{\beta-1} = BS^{\beta-1}$$

This gives rise to a linear consumption rule and a linear extraction rule

$$c = \left[\frac{1}{A(1-\alpha)}\right]^{1/\alpha} K$$

$$r = \left[\frac{A(1-\alpha)}{B}\right]^{1/(1-\beta)} S$$

provided that $A(1-\alpha) > 0$ and $B > 0$. Now we must determine A and B. Let us define

$$w \equiv \left[\frac{1}{A(1-\alpha)}\right]^{1/\alpha} \tag{52}$$

$$\varepsilon \equiv \left[\frac{A(1-\alpha)}{B}\right]^{1/(1-\beta)} \tag{53}$$

Substituting the consumption rule and the extraction rule into the HJB equation (51) we obtain two equations

$$\delta AK^{1-\alpha} = \frac{1}{1-\alpha}w^{1-\alpha}K^{1-\alpha} - A(1-\alpha)wK^{1-\alpha} \tag{54}$$

$$\delta BS^{\beta} = B\left(\frac{A(1-\alpha)-\theta}{B}\right)\varepsilon^{\beta}S^{\beta} - B(\beta)\varepsilon S^{\beta} \tag{55}$$

From equation (54) and using our definition (52), we get

$$\delta A = \left(\frac{1}{1-\alpha} - 1\right)w^{1-\alpha}$$

Hence

$$\frac{\delta}{\alpha}w^{-\alpha} = w^{1-\alpha}$$

Thus

$$w = \left(\frac{\delta}{\alpha}\right)$$

and

$$A = \frac{1}{1-\alpha}\left(\frac{\alpha}{\delta}\right)^{\alpha}$$

For ε to be positive, we need $A(1-\alpha) > 0$. This condition is satisfied .

We can solve for ε and B

$$\varepsilon = \frac{\delta}{1-\beta}$$

$$B = (1-\beta)^{(1-\beta)}\left(\frac{\alpha}{\delta}\right)^{\alpha} > 0$$

Proposition 3 *Assume A1. Under the social planner, the optimal extraction rule and the optimal consumption rule are given by*

$$r = \left(\frac{\delta}{1-\beta}\right)S \quad \textit{and} \quad c = \left(\frac{\delta}{\alpha}\right)K\ .$$

We can now solve for the optimal paths. From

$$\dot{S} = -r = -\left(\frac{\delta}{1-\beta}\right)S \tag{56}$$

we obtain

$$S(t) = S_0 \exp(-\delta t/(1-\beta)) \tag{57}$$

From

$$\dot{K} = \left(K^{\alpha}r^{\beta} - c\right) = \left[K^{\alpha}S^{\beta}\left(\frac{\delta}{1}\right)^{\beta} - \frac{\delta}{\alpha}S\right] \tag{58}$$

and equation (56), we can construct a phase diagram in the space (K, S) where K is measured along the horizontal axis. The locus $\dot{K} = 0$ is given by curve depicting the equation

$$S = \left(\frac{1}{\delta}\right)\left(\frac{\delta}{\alpha}\right)^{1/\beta} K^{(1-\alpha)/\beta} \text{ (this is a locus for } \dot{K} = 0) \tag{59}$$

For points (K, S) above that curve, we have $\dot{S} < 0$ and $\dot{K} > 0$. Below that curve, we have $\dot{S} < 0$ and $\dot{K} < 0$. It follows that the typical optimal trajectory has the shape of an inverted letter C.Starting with a low stock of K, capital at first rises, reaches a peak, then falls.

Given S_0, the dynamic equation for K is, from eq (57) and (56),

$$\dot{K} = K^{\alpha} \left(\frac{S_0 \delta}{1} \right)^{\beta} \exp \left[-\frac{\delta \beta t}{1-\beta} \right] - \frac{\delta}{\alpha} K$$

To solve this equation, let us define

$$\kappa \equiv K^{1-\alpha}$$

then

$$\dot{\kappa} = (1-\alpha) K^{-\alpha} \dot{K} = (1-\alpha) \left(\frac{S_0 \delta}{1} \right)^{1-\beta} \exp \left[-\frac{\delta \beta t}{1-\beta} \right] - (1-\alpha) \frac{\delta}{\alpha} \kappa$$

This is a first order linear differential equation in k of the form $\dot{\kappa} = M \exp(-\lambda t) - D\kappa$, which is easy to solve.

Proposition 4 *Assume the initial conditions* (K_0, S_0) *satisfy the inequality*

$$S_0 > \left(\frac{1}{\delta} \right) \left(\frac{\delta}{\alpha} \right)^{1/\beta} K_0^{(1-\alpha)/\beta} .$$

The optimal path under the social planner consists of two phases. In Phase I, capital is accumulated. In Phase II, both the capital and the resource stocks fall steadily toward zero. Consumption reaches its peak at the transition point between Phase I and Phase II.

Proposition 5 *The value function is*

$$V(K, S) = AK^{1-\alpha} + BS^{\beta}$$

$$= \frac{1}{1-\alpha} \left(\frac{\alpha}{\delta} \right)^{\alpha} K^{1-\alpha} + \left(\frac{1-\beta}{\delta} \right)^{1-\beta} \left(\frac{\alpha}{\delta} \right)^{\alpha} S^{\beta}$$

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