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Abstract

We study the economic mechanism which sustains the substitution of a marginal method for another when demand increases, in the presence of scarce resources. In those Ricardian dynamics, it is shown that the outgoing method is determined by the quantity side of the problem, the incoming method by the value side. That discrepancy explains both the possible failure of the dynamics and the possible occurrence of multiple equilibria. Conditions for existence, uniqueness and the working of the dynamics are stated. A parallel is drawn with the parametric Lemke algorithm used to solve linear complementarity problems.

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1 The Ricardian dynamics

Ricardo (1817) examined the long-run dynamics of capitalism when land was the main scarce resource. With the employment of more and more workers, the production of wheat must be increased and, in a closed economy, less fertile lands are cultivated. The extension of cultivation requires a rise in the price of corn, which must be high enough to make the investment on the lower quality of land as profitable as in any other sector of the economy. Then, the owners of the more fertile lands are in a position to demand a rent of their farmers. The level of the rent on lands of higher qualities, the first to be cultivated, uniformizes the production costs with those on the marginal land. An alternative to the extension of cultivation is its intensification, which consists in using more expensive but more productive methods on the same land, but that possibility does not affect the economic laws.

Under the retained hypotheses, which ignore technical progress, Ricardo's description of the dynamical process is deemed to be well understood, correct and general. Even if agriculture is no longer a strategic sector in many contemporary economies, an expansion of the analysis may consist in studying how the marginal factors shift over time, as a consequence of a shift in the households' demand between closely substitutable products. Our aim is more modest, or more basic, and consists in showing that the analytical grounds of the canonical Ricardian dynamics are not so clear, even in the simplest cases: these difficulties are illustrated in Section 3 by means of a numerical example involving a unique agricultural product and a unique land for which the dynamics fail.

We therefore return to Ricardo's original framework and study it with the help of modern tools. We consider a sequence of long-run equilibria generated by an exogeneously changing final demand basket. The scarce resources are called lands. Time is not taken into account explicitly because the adjustment of prices is presumed to be almost instantaneous. What makes the specificity of the analysis with regard to comparative statics is the following observation: when a change of demand leads to a physical limit on some land, rents and prices jump to a higher level, therefore the value side of the dynamics is discontinuous; by contrast, the scarcity of wheat is met by operating one new method, be it on the same or another land, and that method is introduced at a zero activity level, so that the dynamics of activity levels are smooth. We shall see that this condition suffices to define the sequence of equilibria in a unique way.

Following Sraffa (1960), it is assumed that the independent distribution variable is the given rate of profit: this departure from Ricardo's own hypothesis, who assumed a given real wage, is inessential and leads to analytical simplifications. The retained formalization is fairly general, as it allows for the joint use of several scarce resources and the joint production of several commodities by each method of production (but we consider generic cases only). The notion of long-term equilibrium for a given demand is recalled in Section 2. Section 3 explains how the methods succeed one another when demand changes and why the dynamics may fail. Section 4 states an algebraic condition for their working as well as for local and global uniqueness. Section 5 draws a close parallel between Ricardo's approach and a modern algorithm used to solve linear complementarity problems. Eventually, some references are given in Section 6.

2 Long-term equilibria

The data of a long-term equilibrium are the methods of production, the available areas of lands and the (nonnegative and uniform) rate of profit; its unknowns are the activity levels, the prices and the rents. We write down the conditions defining an equilibrium corresponding to an exogenously given final demand basket d.

Let there be n commodities, m methods of production, g grades of lands, and homogenous labor. Method i ($i = 1, ..., m$) is represented by a input vector $a_i \in R^n_+$ of reproducible commodities, the areas $\Lambda_i \in R^g_+$ of land(s) used and an amount $l_i \in R_+$ of labor, while the output vector is a basket $b_i \in R^n_+$. Production takes one period. Constant returns prevail and the unit activity level of each method is arbitrary. Land is not explicitly considered as a joint output of production. Matrices A, Λ, B are obtained by stacking these vectors, method i being represented by the ith columns of these matrices and scalar l_i. A peculiarity of the formalization here retained is that free disposal and fallowing are listed among the methods (hence, $m > n + g$): the input vector of free disposal is an amount of some commodity, that of fallowing an amount of land, and in both cases the output is zero.

The physical side of an equilibrium states that, for an adequately chosen semipositive vector y of activity levels ($y \in R^m_+$), the demand basket is obtained as the net product of the economy

$$(B - A)y = d \tag{1}$$

and that the scarcity constraints on lands are met. Let vector $\overline{\Lambda} \in R^g_+$ denote the available areas of lands. As fallowing is taken into account into explicitly, the scarcity constraints are written as the vector equality

$$\Lambda y = \overline{\Lambda} \tag{2}$$

The positive components of vector y correspond to the operated methods. The value side of an equilibrium states that any operated method i yields the ruling rate of profit r ($r \geq 0$) and that the non-operated methods do not pay extra profits. For the operated methods, the condition is written

$$b_i^T p = (1+r)a_i^T p + \Lambda_i^T \rho + l_i \tag{3}$$

where p is the price vector and ρ the rent vector (these vectors are nonnegative). labor is chosen as numéraire and the wage $w' = 1$ is paid *post factum* (Sraffa's hypothesis). If fallowing is operated on some land, i.e. if that quality of land is not fully cultivated, equality (3) implies that the rent on that land is zero.

As vector y must satisfy the $n+g$ constraints (1) and (2), the number of operated methods is at least equal to $n+g$, flukes apart. But since a price equality with n prices and g rents holds for each operated method, it is expected that their number does not exceed $n+g$. On the whole, the number of operated methods is equal to $n+g$, flukes apart. If the operated methods are known, solving the system (1)-(2)-(3) determines the activity levels and the price-and-rent vector. Therefore, the only difficulty is to identify the set of operated methods.

Two comments are in order:

- It is more usual to define an equilibrium by means of inequalities with complementarity relationships. Since both definitions are equivalent, the choice between them is a matter of convenience, and the alternative definition will be used in Section 5. The main advantage of the above definition is the constancy of the number of operated methods (including free disposal and fallowing) independently of the demand vector: the extension of cultivation is seen as a substitution of method, possibly for fallowing. With the alternative definition, the number of operated methods varies with demand and an extension of cultivation is seen as the introduction of a new method.

- We ignore degeneracies, and for instance only one land becomes fully cultivated at a given time. Squareness, with $n+g$ operated methods, is a generic property, but an exception to the rule plays a major role in the dynamics: at the point of transition between consecutive equilibria, the activity level of some previously operated method drops to zero and, then, the number of operated methods falls to $n+g-1$.

3 Exit and entry rules

The dynamic approach looks at the evolution of a long-term equilibrium when demand $d = d(t)$ changes with time and proceeds by studying the effect of a change of demand on the set of operated methods. Starting from a given long-term equilibrium E_0, a small change in $d(t)$ triggers the dynamics. As long as lands of the same grade as those presently cultivated are available, the adjustment to demand only requires that of the positive activity levels, with no changes in the operated methods, prices and rents. The physical limit of equilibrium E_0 is reached at $t = t_0$ when the activity level of some operated method vanishes. (Simple as it may seem, the rule lies on the convention that the non-cultivated part of a land is deemed to be 'cultivated' by means of the fallowing method.) Flukes apart, only one activity level vanishes at t_0. That limit defines an exit rule in the sense that it allows us to identify the unique method which belongs to equilibrium E_0 at $t = t_0 - \varepsilon$ and will be excluded from the next equilibrium E_1 at $t = t_0 + \varepsilon$.

At the breaking point t_0, the price of the scarce agricultural good rises up to the level where the use of a more costly method, either on a new type of land (extensive cultivation) or on the same land (intensive cultivation), becomes profitable. A new equilibrium is found and another quiet period with adaptations of positive activity levels opens. The identification of the incoming method relies on the value side of the problem. Beyond the rises in the price of corn and in rents, the whole set of prices is also disrupted because the change in the value of corn modifies the prices of all commodities in the production of which corn enters directly or indirectly. Let us denote the price-and-rent vector (p, ρ) as π. Since we already know all but one of the new operated methods, and therefore all but one of the price equalities (3), the new price-and-rent vector π_1 is known up to one degree of freedom. At this stage, the hypothesis of a given rate of profit leads to a formal simplification: since the price-and-rent equations (3) form a set with as many affine equations as unknowns, the solution to all but one equations is written $\pi_1(\lambda) = \pi_0 + \lambda\pi'$, where vector π_0 is the previous price-and-rent vector, vector π' represents the direction of the change in the price-and-rent vector, and λ is a still unknown scalar representing the intensity of the change.

We can go further and determine the exact value of λ, and therefore the incoming method and the new (potential) equilibrium. The property we use is that the profitability of any method at prices $\pi_0 + \lambda\pi'$ is a monotonous function of λ. For the $n + g - 1$ operated methods which are common to the previous and the new equilibrium, that profitability is constant and measured by the ruling rate of profit. For all other methods, it changes with λ. Consider first the excluded method. As it yields the normal rate of profit at prices π_0, i.e. for $\lambda = 0$, its profitability at prices $\pi_0 + \lambda\pi'$ is either above or below the normal level, depending on the sign of λ. Since the very notion of equilibrium requires that no method pays extra profits at prices π_1, the property determines the sign of λ. Without loss of generality, we assume that λ is positive (otherwise, replace π' by $-\pi'$ and λ by $-\lambda$). Consider now the methods which were not operated in the initial equilibrium, and let the value of λ increase progressively up from zero. At $\lambda = 0$, these methods incur extra costs. Since their profitability varies monotonously with λ, it is expected that, for λ great enough, at least one of them becomes profitable (the existence of such a method is dealt with in Lemma 1 below). The entry rule is: when λ increases, the new operated method is the *first* which yields the ruling rate of profit. For, if λ continued to increase beyond that minimum level, that first method would pay extra profits.

The exit and entry rules determine the evolution of the set of operated methods and define the dynamics: the exit rule relies on the physical side of the model (some activity level vanishes), the entry rule on its value side (some method becomes profitable). The coexistence of rules obeying different principles explains the difficulties met by the dynamics: it is not ensured that the incoming method meets the evolution of demand at the origin of the loss of the previous equilibrium. An illustration is given in a one-commodity model with homogenous land.

Example.

The given rate of profit is $r = 1$. The total area of land is $\overline{\Lambda} = 100$, with three available methods:

(method A) 0.4 qr. corn + 1 acre land + 1 labor $\rightarrow$ 1 qr. corn

(method B) 0.2 qr. corn + 2 acres land + 4 labor $\rightarrow$ 1 qr. corn

(method C) 0.4 qr. corn + 0.25 acre land + 2 labor $\rightarrow$ 1 qr. corn

Let the final demand d increase. For low levels of demand, land is partially cultivated by means of method A, the cheapest method for a zero rent. In that equilibrium E_0, and for a wage 1 paid *post factum*, the price of corn is $p_0 = 5$. Land is fully cultivated at level $d_0 = 60$. For $r = 1$, the general solution to the system (here reduced to the unique price equation attached to method A) $(1 + r)0.4p_1 + \rho_1 + 1 = p_1$ is $\pi_1 = \pi_0 + \lambda\pi'$ with $\pi_1 = (p_1, \rho_1)$, $\pi_0 = (5, 0)$ and $\pi' = (5, 1)$. The profitabilty of both methods B and C improves with λ, and the minimum rule designates method B as the incoming method ($\lambda = 1$), hence $\pi_1 = (10, 1)$. But method B is less productive per acre than A, therefore the progressive substitution of B for A leads to a reduction of the net product, from 60 to 40 quarters after full substitution. The combination of methods A and C, which would solve the physical problem, is excluded because method B would pay extra profits at the associated price-and-rent vector.

Note that the combination of methods (A, B) defines an equilibrium E_1 for any demand between 40 and 60. The phenomenon is general: when a change of method occurs at $t = t_0$, either the new set of methods meets demand $d(t_0 + \varepsilon)$, and then the dynamics work locally, or the previous and the new sets of operated methods constitute two neighboring equilibria, i.e. which differ by one method only, both sustaining demand $d(t_0 - \varepsilon)$.

The numerical example has an additional feature, as there exists an equilibrium E_2 which sustains the production of more than 60 quarters: it is made of methods B and C operating jointly at the price-and-rent vector $(p_2 = 60, \rho_2 = 4)$. Therefore, the sequence E_0, E_1, E_2 of consecutive equilibria connects low to high levels of demand. This path, however, is not economically admissible because the intermediate equilibrium assumes a decrease in the level of demand, whereas a direct jump from E_0 to E_2 at $d = 60$ would imply a discontinuity in activity levels.

4 Local and global dynamics

In this Section, we state the conditions ensuring a smooth physical transition between consecutive equilibria.

Matrix $(B - A, -\Lambda)$ has $n + g$ rows and m columns. At equilibrium E_0, let C_0 be the square matrix of dimension $n + g$ extracted from the columns of $(B - A, -\Lambda)$ and corresponding to the set M of the presently operated methods, i.e. to the positive components y_0 of y, and let δ be the vector obtained by stacking d and $-\overline{\Lambda}$. The unique vector equality

$$C_0 y_0 = \delta \tag{4}$$

summarizes both conditions (1) and (2) and defines the physical side of an equilibrium. On the value side, let $\pi_0^T = (p^T, \rho^T)$ be the price-and-rent vector at equilibrium E_0 and $C_0(r)$ be the square matrix extracted from the columns of $(B - (1 + r)A, -\Lambda)$ corresponding to the positive activity levels. The price equations (3) are written more compactly as:

$$\pi_0^T C_0(r) = l_0^T \tag{5}$$

When the activity level of some operated method vanishes, the reconstruction of the next equilibrium starts from equalities (5), which remain satisfied for all methods but one. Let us consider the set

$$\mathcal{D} = \{d; \exists y > 0 \;\; d << (B - (1 + r)A)y, \Lambda^T y << \overline{\Lambda}\} \tag{6}$$

(For vectors, notation $x >> 0$ means all its components are positive, $x > 0$ that its components are nonnegative and $x \neq 0$, $x \geq 0$ that the components are nonnegative, which is the case for activity levels, prices and rents.)

Lemma 1 *The existence of an incoming method is guaranteed if the demand vector belongs to $\mathcal{D}$.*

Proof. Since the price-and-rent vectors π_0 and π_1 satisfy the same equalities (5) for all common operated methods, their difference $\pi_1 - \pi_0 = \lambda\pi'$ is a solution to $n + g - 1$ linear equalities: $\pi'^T c_i(r) = 0$ for $i \in M \backslash \{j\}$, where j is the outgoing method. Vector π' is unique up to a factor that we choose in order that $\pi'^T c_j < 0$. Then the existence of a minimum positive scalar λ for which some inequality $\pi_0^T c_k(r) < 0$ is turned into the equality $\pi_1^T c_k(r) = 0$ is guaranteed if inequality $\pi'^T c_k(r) > 0$ holds for some method k. If π' has a negative component, this is the case for some free disposal method or some fallowing method. Assume that π' is semipositive. Vector $d(t_0)$ being produced by the methods $i \in M \backslash \{j\}$ and π' being such that $\pi'^T c_i(r) = 0$ for $i \in M \backslash \{j\}$, we have $\pi'^T \delta(t_0) = \pi'^T(d(t_0), -\overline{\Lambda}) = \pi'^T \sum_{i \in M \backslash \{j\}} y_i(t_0)(b_i - a_i, -\Lambda_i) \geq \pi'^T \sum_{i \in M \backslash \{j\}} y_i(t_0)(b_i - (1 + r)a_i, -\Lambda_i) = \sum_{i \in M \backslash \{j\}} y_i(t_0)\pi'^T c_i(r) = 0$. As $d(t_0)$ belongs to $\mathcal{D}$, hypothesis (6) implies the existence of a semipositive vector y such that $C(r)y >> \delta(t_0)$, therefore $\pi'^T C(r)y > \pi'^T \delta(t_0) \geq 0$ and there does exist a method k such that $\pi'^T c_k(r) > 0$. ■

Up from now, it will be assumed that the demand basket belongs to or varies in $\mathcal{D}$. The outgoing and the incoming methods have the following property, which results from the notion of equilibrium: at prices π_0, the outgoing method j yields the ruling rate of profit, whereas the incoming method k is too costly. At prices π_1, the situation is reversed. That alternative admits an algebraic expression:

Lemma 2 $\det C_0(r)$ *and* $\det C_1(r)$ *have opposite signs.*

Proof. Let $\beta_j = (1+r)p_1^T a_j + \rho_1^T \Lambda_j + l_j - p^T b_j = l_j - \pi_1^T c_j(r) > 0$ be the extra costs incurred by the outgoing method j at the new equilibrium prices $\pi_1 = (p_1, \rho_1)$, and $\beta_k > 0$ those incurred by the incoming method k at the previous prices π_0. From equalities $\pi_1^T c_j(r) = l_j - \beta_j = \pi_0^T c_j(r) - \beta_j$ and, similarly, $\pi_0^T c_k(r) = \pi_1^T c_k(r) - \beta_k$, there follows $(\pi_1^T - \pi_0^T)(\beta_k c_j(r) + \beta_j c_k(r)) = 0$. As $\pi_1^T c_i(r) = \pi_0^T c_i(r) = l_i$ for the other $n + g - 1$ common operated methods $i \in M \backslash \{j\}$, and therefore $(\pi_1^T - \pi_0^T)c_i(r) = 0$, it turns out that the matrix with columns $c_i(r)$ and $\beta_k c_j(r) + \beta_j c_k(r)$ has a zero determinant. Therefore $\beta_k \det C_0(r) + \beta_j \det C_1(r) = 0$ and these determinants have opposite signs. ■

Consider now the physical side and suppose that the dynamics work. For $\delta = (d(t_0 + \varepsilon), -\overline{\Lambda})$, a similar alternative holds: at equilibrium E_0, equation

(4) does not admit a solution with positive activity levels, whereas it has a solution at equilibrium E_1. The algebraic expression of the productivity condition is similar to that of Lemma 2, matrix $C_i(r)$ being replaced by $C_i = C_i(0)$:

Lemma 3 *The dynamics work locally if and only if* $\det C_0$ *and* $\det C_1$ *have opposite signs.*

Proof. $\delta(t)$ being the vector obtained by stacking vectors $d(t)$ and $-\overline{\Lambda}$, the physical constraints on activity levels are written as $Cy = \delta(t)$. For $t = t_0 - \varepsilon$, all activity levels y_i $(i \in M)$ are positive at equilibrium E_0 but $y_j(t)$ vanishes at $t = t_0$ and would become negative at $t = t_0 + \varepsilon$. When that method j is replaced by another method k, algebraic decompositions of vector $\delta(t)$ lead to the formal equalities

$$\delta(t) = \sum_{i \in M \backslash \{j\}} y_i(t) c_i + y_j(t) c_j = \sum_{i \in M \backslash \{j\}} y_i'(t) c_i + y_k'(t) c_k \tag{7}$$

At $t = t_0$ both decompositions coincide, with $y_i(t_0) = y_i'(t_0) > 0$ $(i \in M \backslash \{j\})$ and $y_j(t_0) = 0$, therefore $y_k'(t_0) = 0$ (the incoming method starts at a zero level). At $t = t_0 + \varepsilon$, $y_i'(t)$ remains positive by continuity $(i \in M \backslash \{j\})$ and $y_k'(t)$ has a small nonzero value. The new set of methods including method k sustains the demand vector $d(t_0 + \varepsilon)$ if and only if $y_k'(t)$ is positive whereas $y_j(t)$ has become negative at $t = t_0 + \varepsilon$. Equality (7) shows that the $n + g - 1$ vectors c_i for $i \in M \backslash \{j\}$ and vector $y_j(t) c_j - y_k'(t) c_k$ are linearly dependent, therefore $y_j(t) \det C_0 - y_k'(t) \det C_1 = 0$. $y_j(t)$ and $y_k'(t)$ have opposite signs if and only if it is the case for the two determinants. ■

Lemmas 2 and 3 show that the dynamics work locally if and only if the following condition (E) holds:

(E) The sign of $\det C_0(r) / \det C_0$ is preserved when the outgoing method is replaced by the incoming method.

To obtain a global uniqueness result, note first that nothing in the formal arguments prevents that d has negative components (but $d = 0$ is a degeneracy since it is obtained by means of less than $n + g$ operated methods). Let us set hypothesis (H):

(H) Every method other than free disposal and fallowing requires labor.

(For existence results, it suffices to assume that labor is directly or indirectly required to produce a semipositive net product.)

Lemma 4 *Under assumption (H), the equilibrium is unique for any* $d << 0$.

Proof. Free disposal and fallowing sustain the production of $d << 0$, the associated price-and-rent vector being $\pi = 0$. Conversely, the value dp of the net product is at least equal to the wages and to the rents. For $d << 0$, this implies that $p = \rho = 0$ and that the operated methods do not use labor, therefore they are free disposal and fallowing. ■

If condition (E) holds everywhere, a global existence and uniqueness result is obtained:

Theorem 1 *For a given rate of profit, let demand change in $\mathcal{D}$. The dynamics work locally if and only if condition (E) holds. If condition (E) holds everywhere in $\mathcal{D}$, every demand basket d in $\mathcal{D}$ is sustained by a unique equilibrium.*

Proof. Suppose that the sign condition holds for any pair of consecutive sets. Given any two baskets d_0 and d_1 in the convex set $\mathcal{D}$, let us link them by an oriented curve $d(t)$ in $\mathcal{D}$. An equilibrium at d_0 is progressively transferred, by means of successive transforms along the curve, to another at d_1. If d_0 is sustained by a unique equilibrium, could another curve joining d_0 to d_1 lead to a different equilibrium at d_1? Along the curve, an equilibrium is transformed into a uniquely defined neighboring equilibrium when some activity level vanishes. The process is reversible in the sense that, if one starts from the second equilibrium and moves in the opposite direction on the same curve, its successor is the initial equilibrium (because, by construction, the equilibrium which succeeds the second equilibrium is sustained by a price-and-rent vector of the type $\pi_1 - \lambda\pi'$, and the minimality of λ implies that this vector is π_0). Imagine that demand d_1 is sustained by several equilibria. By following the reverse curve from d_1 to d_0, each of them is transferred to d_0 and two equilibria never merge during the successive transforms: otherwise, when moving in the opposite direction, an equilibrium would have two successors. As multiple equilibria at d_1 would give birth to multiple equilibria at d_0, the uniqueness property at d_0 ensures uniqueness for any demand basket. To sum up, if condition (E) holds everywhere in $\mathcal{D}$, uniqueness for some non degenerate demand basket d_0 implies global uniqueness. Lemma 4 concludes. ■

An alternative statement of Theorem 1 is that the multiplicity of equilibria for a given demand d is due to a failure of the dynamics somewhere on a curve leading from low levels of demand to d. It also follows from Theorem 1 that the dynamics always work when the rate of profit is zero. The property is due to the duality between the physical and the value sides: for a zero rate of profit, the profitability criterion and the productivity criterion coincide (golden rule).

5 From Ricardo to Lemke

One and a half century after Ricardo, mathematicians elaborated algorithms to find a solution to a class of problems called linear complementarity problems. As it turns out that the static (i.e., for a given demand vector) long-term equilibrium is of that type, we study the relationships between the Ricardian dynamics and an algorithmic approach of the Lemke type.

We now refer to the alternative definition of an equilibrium mentioned at the end of Section 2: an equilibrium is a nonnegative solution (p, ρ, y) to the system of vector inequalities with complementarity relationships

$$
\begin{aligned}
(B-A)y - d &\geq 0 \qquad [p] && (8)\\
-\Lambda y + \overline{\Lambda} &\geq 0 \qquad [\rho] && (9)\\
((1+r)A - B)^T p + \Lambda^T \rho + l^T &\geq 0 \qquad [y] && (10)
\end{aligned}
$$

where the variables into brackets are componentwise complementary to the associated inequalities: they are nonnegative, and zero if the corresponding component of the inequality is strict. The gap between (8)-(9) and the previous formalization (1)-(2) of an equilibrium is only apparent: if a commodity is in excess supply, its price is zero, the free disposal method yields the ruling rate of profit and its use allows us to switch from inequality (8) to equality (1); the same for land.

Let w be the vector obtained by stacking the left-hand sides of inequalities (8)-(9)-(10), i.e. the excess supply, the non-cultivated areas and the extra costs; let z be the vector of complementary unknowns obtained by stacking the price vector p, the rent vector ρ and the activity levels y; let q be the vector of data obtained by stacking $-d, \overline{\Lambda}$ and l^T; finally, let M be the square matrix

$$
M = \begin{bmatrix} 0 & 0 & B-A \\ 0 & 0 & -\Lambda \\ (1+r)A - B & \Lambda^T & 0 \end{bmatrix}
$$

A long-term equilibrium is a solution (z, w) of the system

$$
\begin{aligned}
w &= Mz + q && (11)\\
w &\geq 0, z \geq 0, w_i = 0 \text{ or } z_i = 0 \text{ for any } i && (12)
\end{aligned}
$$

That problem is called a linear complementarity problem and is denoted LCP(q, M). Not all LCPs have solutions, and different techniques have been elaborated to find one of them, if any. Note that, if one knows the set S of the zero components of z, the complementary set $\overline{S}$ is made of zero components of w, and all other components of z and w are easily found by solving a square system of linear equations, which is assumed to be nondegenerate. The core of the problem is therefore to identify the distribution of zeroes between z and w. (Similarly, if one knows the set of non-operated methods, all characteristics of an equilibrium are easily found.)

Lemke and Howson (1964) conceived an algorithm to find a solution to the bimatrix game. Interestingly enough, the formalization of a bimatrix game as a linear complementarity problem lets appear a matrix M with two blocks of zeroes on the diagonal, as in the long-term equilibrium problem (but the vectors q differ). Lemke (1965) generalized the procedure, and many variants were introduced later. The problem is to identify the set S. The general principle of a Lemke algorithm is to start from an 'almost solution' S which suffers from some defect, then to improve it by changing one of its elements, then another, according to a mechanically determined sequence of trials, until the elimination of the defect. An important property of the procedure is that it does not return to an already examined set S. Since the number of sets is finite, the algorithm finds a solution or collapses at some stage (which is the necessary outcome if no solution exists). A variant of that algorithm is the parametric Lemke algorithm (Cottle *et al.* 1992): its basic idea is to determine a solution of LCP(q_1, M) by transferring a known solution of LCP(q_0, M) along a curve $q(t)$ joining q_0 to q_1. The initial defect is that the almost solution is a solution to LCP(q, M), but for a wrong value of q. A solution at q_0, more generally at $q(t)$, is defined by the zero $S-$components of z, the zero $\overline{S}-$components of w and the other positive components. Slight changes in $q(t)$ are met by adjusting these positive components, with no change in the set S. The adjustment stops when some positive component of z or w, say $w_j(t)$, vanishes at some $t = t_0$. Since $w_j(t)$ was positive at $t_0 - \varepsilon$, index j is in S. When $q(t)$ continues to evolve, the next set S' is obtained by changing the set S and transferring index j from S to $\overline{S}$: the constraint $w_j = 0$ is now substituted for the previous one $z_j = 0$. In a neighborhood of t_0, the other positive components of $(z(t_0), w(t_0))$ remain positive by continuity.

It turns out that, from a formal point of view, the parametric LCP algorithm is a version of the Ricardian dynamics in which vector $q(t_0)$ plays the role of an augmented demand vector. In general, the transfer from $q(t)$ to $q(t + \varepsilon)$ sets no problem, with a rupture from time to time when a positive variable drops to zero. The new set S defined by the algorithm is a neighbor of the previous, as only one nonzero variable is changed. In the literature on LCP, the search for a new positive component or a new facet is similar to that of the incoming method as described in Section 3 and the choice is also determined by a minimum rule (Lemma 1 simplifies the problem as it solves all difficulties linked to existence).

There is, however, one significant difference. It has been noticed that the incoming method may be less productive than the outgoing method and, then, the Ricardian dynamics stop. A similar difficulty is met in the LCP framework: if z_k is the new variable which was zero and becomes nonzero, equality $w = Mz + q(t)$ holds at $t = t_0$ with $z_k(t_0) = 0$, while the positive components of (w, z) satisfy complementarity conditions. At $t = t_0 + \varepsilon$, these variables remain positive by continuity, but the algebraic decomposition $w = Mz + q(t_0 + \varepsilon)$ only ensures that the value of $z_k(t_0 + \varepsilon)$ is close to zero. If that component is positive, the process leads to a transfer of the solution from $q = q(t_0 - \varepsilon)$ to $q = q(t_0 + \varepsilon)$. If, on the contrary, $z_k(t_0 + \varepsilon)$ is negative, then $z_k(t_0 - \varepsilon)$ is positive and a second solution to LCP$(q(t_0 - \varepsilon), M)$ is found. By contrast with the Ricardian dynamics, the Lemke algorithm goes on by making a U-turn on the curve. The algorithm continues for lower and lower values of t ('antitone' move, in the terminology of the LCP literature). Eventually, however, another U-turn may occur (or must occur, if some condition ensuring the working of the parametric algorithm is met) and new solutions sustaining increasing levels t are obtained. A third solution sustaining $q(t_0 - \varepsilon)$ is first reached and, finally, a solution for level q_1.

The point is illustrated by the discussion of the example in Section 3: for low levels of demand, method A is the cheapest (equilibrium E_0). At level $d_0 = 60$, the land is fully cultivated and the dynamics stop because the next cheapest method is less productive. On the contrary, the parametric Lemke algorithm goes on by switching to equilibrium E_1, allowing for a temporary decrease in the net product. Method B is progressively substituted for A, and the net product drops from 60 to 40. At this stage, method C is introduced and the final equilibrium E_2 sustains high levels of demand. As far as one looks at the existence of long-term equilibria in a static framework, not at the dynamics, the following result illustrates the powerfulness of the algorithm:

Theorem 2 *For any d in $\mathcal{D}$, and flukes apart, the number of equilibria is odd, with one more equilibrium for which the signs of* $\det C(r)$ *and* $\det C$ *are the same.*

Proof. Let us draw a curve $d(t)$ in $\mathcal{D}$, without self-intersection, from $d_0 << 0$ to a given vector $d_1 \in \mathcal{D}$. The unique solution at d_0 is transferred along that curve, and the sequence of equilibria is mechanically defined by the entry and exit rules, with a reversibility property. If, for some demand vector, the same equilibrium were found twice, once for increasing values of t and next for a decreasing value, reversibility implies that the same property would hold for the successor of that equilibrium when t increases, and a contradiction is obtained by considering the first equilibrium which is found twice. As the number of equilibria for a given demand basket is finite, the algorithm must reach d_1, in spite of possible U-turns. The conclusion is that the equilibrium at d_0 is ultimately transformed into some equilibrium at d_1.

Let us now draw a curve, without self-intersection, from $d_0 << 0$ to d_1 and coming back to $d_2 << 0$. Consider an equilibrium at d_1. If the transfer of that equilibrium along the curve leads ultimately to either d_0 or d_2, that equilibrium is met somewhere in the unique sequence of equilibria (such a sequence is called a path, to distinguish it from the curve $d(t)$) which, starting from d_0, goes to d_2. That path sustains an equilibrium at d_1 an odd number of times, $s+1$ times for increasing values of t (therefore, with $\det C(r)$ and $\det C$ of the same sign) and s times in the other direction. The alternative is that the path starting from d_1 never reaches d_0 or d_2. Since there are finitely many combinations of $n+g$ operated methods, the path admits a loop. Suppose that the path itself is not a loop. Then, the first combination of methods which is found twice on the path has two predecessors. By the reversibility property, an equilibrium on the reverse path would have two successors, a contradiction. Therefore the path itself is a finite loop. Everytime it reaches d_1 it defines a new equilibrium at that point, with as many equilibria with increasing values of t as with decreasing values. These additional equilibria do not change the balance between the two types of equilibria. ∎

A part of the LCP literature deals with conditions ensuring the absence of antitone move: in the long-term equilibrium framework, these conditions amount to ensuring the working of the dynamics.

6 References to the literature

Ricardo showed that, in normal times, the evolution of demand is met by an adjustment of activity levels with no changes in prices and rents, but prices and rents jump abruptly when a land is fully cultivated, the rise being sufficient to let some non-operated method yield the ruling rate of profit. The adjustment to demand is met by introducing a new method of cultivation on the same (intensive cultivation) or another (extensive cultivation) land. Ricardo failed to notice a third possibility, which consists in introducing a corn-saving method in industry, and was mistaken in assuming that it is always possible to get rid of rent by considering the set of marginal methods (Bidard 2012). Sraffa's (1960) formalization of prices and rents was later completed to take explicitly into account a demand vector and the scarcity constraints. The discovery of the possible multiplicity of equilibria for a given demand (D'Agata 1982), Freni (1991) was a surprise, but the reason of these complications remained obscure because the dynamic approach had been abandoned and replaced by the search of a solution of a given system of inequalities with complementarity relationships. Salvadori (1986) transferred a result from the literature on linear complementarity (Dantzig and Manne 1974) to state existence for any demand basket in the set $\mathcal{D}$ defined by (6), but the algorithm he referred to has no clear economic interpretation. Starting from geometrical considerations, Erreygers (1990, 1995) stated the criterion (E) for local uniqueness and argued that global uniqueness might also be obtained under additional assumptions on the structure of equilibria. Theorem 2 generalizes Bidard and Erreygers's (1998) oddity result for economies without lands. The relative sign of the determinants plays a role similar to the index in the differentiable version of general equilibrium theory (Mas-Colell 1985).

7 Conclusion

When the direct or indirect demand for a resource increases and meets a scarcity limit, the prices rise and the price system adapts itself in such a way that demand is met by means of a new method. That method may either make use of an alternative resource (as exemplified by the notion of extensive cultivation), or may make a more efficient use of the same scarce resource (intensive cultivation), or may save the resource. Ricardo was the first economist to give a rather precise description of the dynamical aspects of the substitution process in the productive system, but a detailed analysis shows that the working of that process is not so clear. The ultimate reason of the difficulty is that prices are not a perfect scarcity index, so that the profitability of a method may be disconnected from its ability to meet the requirements for use. The present study has greatly simplified the dynamics themselves by assuming an instantaneous adjustment and has let aside the analysis of substitutability on the demand side, a phenomenon which plays an important role in actual economies. Increasing substitutabilty either on the supply side by assuming differentiability or on the demand side may favour the working of the dynamics, but the difficulties pointed at in a simple framework should not disappear by complexifying the problem.

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