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a Classical approach

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Disequilibrium, reproduction and money: a Classical approach

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Abstract. We consider a bisector reproduction model in which money is introduced as a pure means of exchange issued by a bank at the producers' requests. Each capitalist aims at maximising accumulation in his own sector. Their plans are based on available quantities and expected prices. Effective prices are determined by a market-clearing mechanism. Temporary disequilibria occur in both physical and monetary terms. The settlement of the monetary balances is operated by means of a transfer of capital goods. Final allocations and effective productions are thus determined. The dynamics of the economy are those of a sequence of temporary disequilibria and let appear several possibilities (local or global stability, cycles) depending on the values of the parameters.

Keywords. Classical Reproduction, Monetary prices, Disequilibrium, Growth, Cycle.

JEL classification: E11, E30, E32, O41

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1 Introduction

The¹ model studied in this paper represents a bisector economy in temporary disequilibrium. It is an extension of a previous model (Benetti *et al.* 2012) inspired by Torrens's (1821, chap. VI, sect. VI) and Marx's (1885, Book II, chap. XXI) approaches of reproduction. Its main innovation is the introduction of monetary exchanges instead of barter: we consider a monetary economy in which money is the unit of account and a pure means of exchange issued by a bank. That issue is made at the producers' requests, and the producers commit themselves to reimburse the bank immediately after the exchange. Money is not a specific good in the initial endowments; it is not either a credit or a store of value and its amount is an endogenous magnitude. That conception of money, though not dominant in economic theory, can be traced back at least to Wicksell (1906), is found in post-Keynesian approaches and is also adopted by Drèze and Polemarchakis (2000) in a general equilibrium framework. Unlike these authors, however, our analysis introduces a market mechanism determining monetary prices both in disequilibrium and equilibrium. Individual decisions are taken independently by each agent, with no *a priori* coordination and therefore the analysis is centred on temporary disequilibrium rather than equilibrium.

The introduction of money does not modify the very notion of equilibrium but allows us to better analyse the working of an economy in disequilibrium. The availability of a means of exchange allows each agent to buy goods independently of his sales. As a consequence, monetary balances appear and must be settled. The agents' disequilibria are both real (they do not fulfil their plans) and monetary (positive or negative balances appear). An institutional rule is introduced and determines the way to ensure the settlement of these monetary balances. Here we adopt a simple rule and assume that the settlement is made by means of a transfer of ownership of some capital goods, chosen by the capitalist with a positive monetary balance, at the expense of the other capitalist. That procedure modifies the allocation of resources as it had been determined previously by the exchange at market prices. The introduction of money as a pure means of exchange thus affects the real dynamics of the economy.

In Section 2 we build up a model which exhibits these features and study its properties. Given the quantities produced during period $t-1$ and before the opening of the market at date t (end of

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period $t-1$ and beginning of period t), each producer asks for and obtains an amount of money equal to the expected value of his supply. A market mechanism determines the monetary prices and the physical allocations which in disequilibrium differ from the expectations. The agent's monetary receipts also differ from his expenses and balances appear. The rule retained for balance settlements reallocates the goods among agents. The final allocation of means of production is then obtained and the effective productions of period t are determined. Section 3 studies the dynamics of the proportions in production and of monetary prices. The dynamics are those of a sequence of temporary disequilibria, not of temporary equilibria since it is only by fluke that the agents fulfil their plans.

2 Real and monetary disequilibrium

2.1 Producers' plans

The hypotheses related to the real part of the model are the same as in Benetti *et al.* (2012):

- Labour does not appear explicitly, every worker being replaced by the corresponding wage basket, which is incorporated in the means of production.
- There are two sectors and one method of production per sector. The technique is represented by matrix $A = [a_{ij}]$ ($i, j = 1, 2$), where a_{ij} is the amount of input j entering into the production of one unit of good i . Matrix A is productive and indecomposable ($a_{12} > 0, a_{21} > 0$).² Constant returns prevail, production takes one period, all capital is circulating and all goods are perishable.
- Goods can be disposed of freely.
- Each capitalist aims at maximising accumulation in his own sector.³
- All capitalists are assumed to have the same price expectations. We may therefore aggregate them and consider only one capitalist per branch.

² The reason of the indecomposability hypotheses is that the accumulation process we study rests on the demand by a capitalist of the input produced by the other. If the two-good economy is non-basic, some agent does not demand the other's good and any exchange is excluded.

³ Maximising accumulation is a quite usual hypothesis in the classical tradition, connected with the idea that accumulation is the best way to get future profits. In this tradition, the hypothesis that capitalists invest in their own sector is found in Torrens (1821) and in Marx's (1885) enlarged reproduction schemes. That hypothesis excludes the existence of a capital market and allows us to isolate the role of money as a pure means of exchange.

At a given date before the opening of the market, producer i ($i=1,2$) has produced a quantity q_i^- and his price expectations are p_1^e and p_2^e (index e is for ‘expected’ and refers to prices; at this stage price expectations are exogenously given). Let q_i^n be his planned level of production (index n is for ‘notional’ and refers to quantities) for the opening period. His expected budget constraint is written $q_i^- p_i^e \geq q_i^n (a_{ii} p_i^e + a_{ij} p_j^e)$: expected proceeds from the sale of output in the just completed production cannot be less than the expected cost of production for the forthcoming period. Accumulation desires being unbounded, this *ex ante* constraint is binding and therefore:

$$q_1^n = \frac{q_1^- p_1^e}{a_{11} p_1^e + a_{12} p_2^e} \quad (1)$$

$$q_2^n = \frac{q_2^- p_2^e}{a_{21} p_1^e + a_{22} p_2^e} \quad (2)$$

Formulas (1) and (2) show that plans only depend on the relative price $p^e = p_1^e / p_2^e$, not on monetary prices. The same for their ratio $q^n = q_1^n / q_2^n$:

$$q^n = q^- p^e \frac{a_{21} p^e + a_{22}}{a_{11} p^e + a_{12}} \quad (3)$$

where $q^- = q_1^- / q_2^-$. In general, the capitalists’ plans are not feasible.

2.2 Money and market prices

Money as the means of exchange is issued by a bank at the agents’ request. The amount of money demanded by each agent is equal to the expected value of his purchases of goods, which itself is equal to the expected value of his supply of goods. Prices are measured in terms of money that the economy acknowledges as the common unit of account. Each agent commits himself to reimburse the money he received from the bank (this is why money does not appear in the *ex ante* budget constraint), and money is destroyed after it has accomplished its circular flow. As we assume that the creation of money, its circulation and its destruction take place within a very short time interval, the rate of interest is ignored. There is neither financing of present purchases by future resources (no credit) nor financing of future purchases by today’s money (money is not considered here as a store of value).

Let us assume that the formation of market prices, denoted p_i , obeys the simple rule initially stated by Cantillon: “Prices are fixed by the proportion between the produce exposed for sale and the money offered for it” (1755, I.IV.2). That rule was also used by Smith and reappears in modern theory of strategic market games (Shapley and Shubik, 1977).

As a consequence of monetary exchanges, two variants of the rule can be considered: (i) the whole production q_i^- is brought to the market; or, (ii) the supply is the difference between q_i^- and the quantity $q_i^n a_{ii}$ retained by producer i as an input for his next production. The latter is the only possibility in a barter economy. In the terminology of strategic market games, the first variant corresponds to the ‘all-for-sale’ model, the second to the ‘offer-for-sale’ model. In the first case, ‘wash sales’ occur as the agents are on both sides of the market. In our model, the distinction has no incidence at equilibrium but matters in temporary disequilibrium when the expected prices and the market prices differ. We consider here the all-for-sale variant in which all produced quantities are evaluated at market prices (the alternative model is briefly studied in Appendix 2).

The amount of money demanded and spent by agent i is $p_i^e q_i^-$. The quantity of money brought on market i for purchases is equal to the expected value of the demand d_i for inputs by both agents, as defined by their plans. The physical quantity s_i brought on that market is the produced quantity q_i^- of that good. The Cantillon rule defines the market price of good i at the current date as the one given by the exchange of these quantities of money and good:

$$p_i = \frac{p_i^e d_i}{s_i} = \frac{p_i^e (a_{ii} q_i^n + a_{ji} q_j^n)}{q_i^-} \quad (4)$$

hence the relative price in terms of good 2

$$p = \frac{p^e a_{11} q_1^n + a_{21}}{q^- a_{12} q_1^n + a_{22}} \quad (5)$$

The application of the Cantillon rule determines the market prices and the post-market allocations of commodities.

In the model we consider money is endogenous. The issued quantity M of money, equal to the total expenses, is based on expected monetary prices ($M = p_i^e q_i^- + p_j^e q_j^-$) and, since M also

represents total receipts, the monetary prices satisfy equality $M = p_i q_i^- + p_j q_j^-$. (If $p_i = p_i^e$ then $p_j = p_j^e$). The endogeneity of money lets the general level of market prices depend on the level of expected monetary prices and explains why the agents correctly foresee monetary prices as soon as they correctly foresee the equilibrium relative price. We denote p_T that relative price (index T is for Torrens).⁴ When the monetary expected prices are in that right proportion, and whatever their level, they are self-fulfilling. Then both markets are in temporary equilibrium: at that relative price and for the period we are considering, the agents' plans are met and both commodities are entirely accumulated. By inserting the relative production plan q^n given by (3) into (5) we get the relative market price as a function of the relative expected price and the relative present quantities. The Torrens price is obtained by setting the equality between p and p^e .

Flukes apart, prices expectations are not fulfilled. Since equality $q_i^-(p_i^e - p_i) + q_j^-(p_j^e - p_j) = 0$ holds in any case, it follows $p_i^e \geq p_i \Leftrightarrow p_j^e \leq p_j$. If the agents' price expectation for good i is greater than its market price, they overvalue that good and undervalue the other good j . Formula (4) shows that $p_i \geq p_i^e \Leftrightarrow d_i \geq s_i$: the undervalued commodity is in excess demand and is called 'scarce', while the overvalued commodity is in excess supply and is called 'superabundant'.

When the agents' price expectations differ from market prices, two types of individual disequilibria appear in spite of the clearing of both markets. The real disequilibrium springs from the agent's role as a buyer: his mistaken expectations lead to his owning a basket of commodities different from the one he had scheduled. The monetary disequilibrium springs from his role as a seller: his expenses differ from his monetary receipts and a monetary balance appears. The individual balance of producer i is positive if good i is scarce, negative if not, and the algebraic sum of the balances is zero.

2.3 Balances settlement and effective production

The agents react to these disequilibria, if only because balances settlements cannot be delayed

⁴ Torrens (1821) emphasized the physical constraints imposed to the reproduction of capital when the profits of each industry are entirely accumulated in the same sector.

since money is considered as a pure means of exchange and not a store of value. How do they react and fulfil their monetary commitments? An institutional rule is here required, which results in a settlement of monetary balances and in post-market changes in the allocation of resources. We now describe a plausible rule of that type.

Without loss of generality assume for the moment that commodity 1 is the one which was undervalued by the agents (with good 2 as numeraire, $p_1^e < p_1$). The agents' positions are asymmetrical:⁵ the producer of the superabundant good, whose balance at the closure of the market is negative, cannot reimburse the advances made by the bank. The goods he owns might be seized by a liquidator. The institutional rule we admit is the following: the producer with negative money balance obtains the money he needs to pay back the bank by transferring some of his real assets to the agent 1 with positive money balance. We assume that the transferred goods are evaluated at market prices and that the physical composition of the transferred basket is determined by producer 1, who is in a position to impose it: he chooses it in order to maximise his next effective production q_1 .

Let us determine the final allocation of resources when that rule is followed. Producer 1's sales amount to $q_1^- p_1$. Since the production cost of every unit of good 1 in the next period is $a_{11}p_1 + a_{12}p_2$, the quantity q_1 that this entrepreneur will produce by investing the whole of his receipts amounts to

$$q_1 = \frac{q_1^- p_1}{a_{11}p_1 + a_{12}p_2} \quad (6)$$

That production is greater than the one he had planned before the market. The existence of a gap between expected and effective quantities is a feature common to all disequilibrium models. The property which can be specifically assigned to the introduction of money and the above balance settlement rule is that the agent with a positive monetary balance succeeds in accumulating his whole effective profits.

After the transfer of the means of production as decided by agent 1, the producer of the

⁵An asymmetric relationship between producers is also found in Marx's enlarged reproduction schemes: the capitalists of sector I (production of means of production) decide how much they accumulate, those of sector II (production of consumption goods) adapt their accumulation to the decision taken by sector I (Marx, 1885, Book II, chap. XXI).

superabundant good 2 owns $(q_1^- - a_{11}q_1)$ units of good 1 and $(q_2^- - a_{12}q_1)$ of good 2. In general, the proportion between these quantities does not fit the technical requirements of industry 2, so that some good will not be entirely used as input. The next effective production of good 2 will amount to

$$q_2 = \min\left(\frac{q_1^- - a_{11}q_1}{a_{21}}, \frac{q_2^- - a_{12}q_1}{a_{22}}\right) \quad (7)$$

Let us note that both agents receive from the market more of the superabundant good than they had planned.⁶ Does the mechanism we consider violate the principle of voluntary exchange (see Bénassy, 1986), which expresses that no agent is obliged to sell or buy more than he had planned at market prices? The question is irrelevant in the present model for two reasons. First, as the agents' plans are based on expected prices while the endogenous market prices are initially unknown, a comparison between their expected and effective positions is irrelevant when these prices differ; and, second, the final allocation results from balance settlements which are of an institutional nature. The agent with a negative monetary balance has no idea about the quantity of the superabundant good he will be compelled to transfer to the other agent. He cannot know if he bought too much or too little of a good. The very notion of voluntary exchange does not fit with that framework.

2.4 Normal and pathological cases

In the economy we study, it is expected that the scarce commodity 1 will be entirely used as input, so that we have $q_1^- = a_{11}q_1 + a_{21}q_2$. By multiplying both sides by p_1 and comparing with relation (6), there comes equality:

$$a_{12}q_1p_2 = a_{21}q_2p_1 \quad (8)$$

This equality tells us that the value of the superabundant commodity used in the production of the scarce commodity equals the value of the latter used in the production of the former. Its noteworthy feature is that both commodities play a symmetric role in that relation: had good 2

⁶ A similar phenomenon holds in general strategic market games. Shapley and Shubik (1977, p. 947) only noticed that "it is a matter of letting one's stomach rather than one's purse absorb the fluctuations".

been the scarce good instead of 1, we would have obtained the same formula. For the remainder of the analysis, and whatever the scarce and the superabundant goods are, we shall choose an arbitrary good, say good 2, as the numeraire for the relative price ($p = p_1 / p_2$) and the relative quantity ($q = q_1 / q_2$). (That convention will be useful in the study of the dynamics when the same good may become alternatively scarce or superabundant.) Equality (8) is then written as:

$$q = \frac{a_{21}}{a_{12}} p \quad (9)$$

This formula shows that the relative quantity depends only on the relative market price.

The production plans $q_1^n(p^e)$ and $q_2^n(p^e)$, as determined by formulas (1) and (2), allow us to calculate the *ex ante* excess demands as a function of p^e . For example the *ex ante* excess demand of good 1 is $a_{11}q_1^n(p^e) + a_{21}q_2^n(p^e) - q_1^-$, and commodity 1 is the *ex ante* scarce commodity if its excess demand is positive. Had the agents known the market prices, the excess demand for commodity 1 would have been obtained by replacing p^e by p in the same expression. The *ex post* scarce commodity is the one in excess demand at market prices: in our model, it is that commodity which is entirely accumulated. Usually, the *ex ante* and the *ex post* scarce commodities coincide, but they may differ in some pathological cases. The normal case occurs when $(p^e - p_T)$ and $(p - p_T)$ have the same sign. It can be shown that a sufficient (though not necessary) condition for normality is written

$$D > -4\sqrt{a_{11}a_{12}a_{21}a_{22}} \quad (10)$$

where D denote the determinant of the matrix A of technical coefficients. Hence, the normal case always occurs if the determinant is positive or slightly negative. In the pathological case, formula (9) does not hold and the study of the dynamics is more complex. A viability problem may even appear if the producer of the *ex ante* scarce commodity cannot accumulate all his profits because the other good is in short supply and, then, a modification of the institutional setup assumed above is necessary. For the remainder of the analysis, we shall assume that condition (10) is met.

3 Dynamics

3.1 Real dynamics of disequilibria

We have analyzed the behaviour of the model within a period, given the just completed productions and price expectations. The model determines the market prices and, after the balance settlements, a new allocation of inputs between the agents. A hypothesis on the formation of expectations allows us to make the link between these market prices and the expected prices for the next period, and therefore to define the dynamics of the model.

Inserting (3) in (5) gives the relative price p as a function of q^- and p^e . To define the dynamics we assume that the expected market price at some date is the effective price at the previous date: $p_i^e = p_i^-$. (This static expectations hypothesis is only retained for its simplicity.) Since we are studying dynamics, the time index t is now written down explicitly, so that this relation is written as:

$$p_{i,t}^e = p_{i,t-1}^-. \quad (11)$$

When the proportion is that of the Perron-Frobenius row-eigenvector $(q^*, 1)$ of matrix A and the price expectations are the corresponding column-eigenvector, all goods are accumulated, the price and quantity expectations are met and the economy follows a regular growth path at maximum rate (von Neumann growth rate). Otherwise, the proportion varies from a period to the next, there are balance settlements and some goods are excluded from accumulation. As goods which are scarce during some period are superabundant during other periods, relation (9), which holds independently of the nature of the scarce good, plays an essential role in the study of the dynamics of the relative quantity.

The evolution of the effective relative quantity q between consecutive periods can be studied by considering the planned relative quantity q^n . On the one hand, formula (3) expresses the relative quantity planned at date t as a function of the previous produced quantity and the expected relative price, equal to the previous market price. That market price also results from the previous quantity by means of equality $p^- = \frac{a_{12}}{a_{21}} q^-$ (according to relation (9) applied at date $t-1$). On the whole, with explicit time indices, the relation between the planned quantities at date t and those

produced at date $t-1$ is written:

$$q_t^n = q_{t-1} p_{t-1} \frac{a_{21} p_{t-1} + a_{22}}{a_{11} p_{t-1} + a_{12}} = q_{t-1}^2 \frac{a_{12} q_{t-1} + a_{22}}{a_{11} q_{t-1} + a_{21}} = f(q_{t-1}) \quad (12)$$

On the other hand, formula (5) gives the relative price (and therefore the relative quantity, as a consequence of relation (9)) as a function of the expected price and the planned production. With static expectations, one obtains:

$$q_t = \frac{a_{11} q_t^n + a_{21}}{a_{12} q_t^n + a_{22}} = g(q_t^n) \quad (13)$$

The combination of formulas (12) and (13) defines the dynamics of q_t . The explicit formula

$$q_t = \frac{a_{11} a_{12} q_{t-1}^3 + a_{11} a_{22} q_{t-1}^2 + a_{21} a_{11} q_{t-1} + a_{21}^2}{a_{12}^2 q_{t-1}^3 + a_{12} a_{22} q_{t-1}^2 + a_{11} a_{22} q_{t-1} + a_{22} a_{21}} \quad (14)$$

is far from being attractive, but the economic phenomena at stake become clearer when one keeps in mind the retained decomposition:

- the planned relative quantity increases with the actual relative quantity at the previous period (formula (12));
- the actual relative quantity increases with the planned relative quantity if $D > 0$, and decreases if $D < 0$ (formula (13)). The sign of D admits an economic interpretation: it is positive if a_{11}/a_{12} is greater than a_{21}/a_{22} , i.e. if each industry makes a relative greater use of its own product, and negative in the other case.

Clearly enough, the unique stationary point q^* of the dynamics (14) corresponds to the proportion of the eigenvector vector $(q^*, 1)$ as q^* is preserved by functions f and g . Moreover, since function g is bounded by a_{11}/a_{12} and a_{21}/a_{22} , the dynamics defined by (14) are never explosive. But several evolutions are possible: it is shown in Appendix 1 that the dynamics depend on the sign of D : If D is positive, the system converges towards a von Neumann growth path. If D is negative, either that convergence is local (and may be global), or there exists a limit cycle of order two, depending on the ratio between the second and the first (or dominant) eigenvalue of matrix A .

Thanks to relation (9), the dynamics of the relative price p are basically the same as those of the relative quantities. In particular, in case of convergence, the relative price tends towards the right

eigenvector $(p^*, 1)$ of matrix A , i.e. towards the price of production as defined by the Classics.

3.2 Nominal dynamics

Let us consider the normal case with convergence towards equilibrium. According to relation (9), the dynamics of the relative quantity and the relative price are identical. We look here at absolute magnitudes, viz. the produced quantities of each good and the nominal prices.

Rather unexpectedly, there exists an asymmetry between the dynamics of quantities and those of prices. The rule adopted for quantities makes reference in equation (7) to function *min* and is not amenable to a simple analytical study. By contrast, the evolution of nominal prices is much more regular. By substituting the agents' plans (1) and (2) in equalities (4), the static expectations hypothesis (11) leads to the induction formulas:

$$\frac{p_{1,t}}{p_{1,t-1}} = \frac{a_{11}p_{t-1}}{a_{11}p_{t-1} + a_{12}} + \frac{a_{21}}{(a_{21}p_{t-1} + a_{22})q_{t-1}} \quad (15)$$

$$\frac{p_{2,t}}{p_{2,t-1}} = \frac{a_{12}q_{t-1}p_{t-1}}{a_{11}p_{t-1} + a_{12}} + \frac{a_{22}}{a_{21}p_{t-1} + a_{22}} \quad (16)$$

(where $\frac{q_{1,t}}{q_{2,t}} = q_t$ and $\frac{p_{1,t}}{p_{2,t}} = p_t$). The magnitudes on the left-hand sides of these equalities represent the inflation factors between dates $t-1$ and t . It turns out that changes in monetary prices are explained by real factors (previous relative price and previous relative quantity).

By using (9) to eliminate q_{t-1} , the same relationships define the evolution of nominal prices by the induction formulas:

$$\begin{cases} p_{1,t} = \frac{a_{11}p_{1,t-1}^2}{a_{11}p_{1,t-1} + a_{12}p_{2,t-1}} + \frac{a_{12}p_{2,t-1}^2}{a_{21}p_{1,t-1} + a_{22}p_{2,t-1}} \\ p_{2,t} = \frac{a_{21}p_{1,t-1}^2}{a_{11}p_{1,t-1} + a_{12}p_{2,t-1}} + \frac{a_{22}p_{2,t-1}^2}{a_{21}p_{1,t-1} + a_{22}p_{2,t-1}} \end{cases} \quad (17)$$

A doubling of nominal prices at date 0 leads to the same proportional change at any date. In the case of convergence, the nominal prices at equilibrium are also doubled. That is why, if one tries to find directly the long-term stationary prices from equations (17), it turns out that these two equations reduce to one, whose solution is the relative price p^* as given by the Perron-Frobenius eigenvector of matrix A .

Consider now the effects of an exogenous non proportional shock on expected monetary prices, when the economy is assumed to be at equilibrium at the initial date, all goods being entirely accumulated. As the relative expected price is changed, the agents' demands are modified: some good becomes superabundant and is not totally accumulated; monetary imbalances also appear, which require monetary settlements. Effective productions, as well as their proportion, are modified. The economy is thus submitted to real and monetary disequilibria. In particular, the goods excluded from accumulation do not contribute to economic growth. This phenomena last all along the transitional dynamics until a new equilibrium is reached in the long run. The economy then recovers its initial von Neumann rate of growth (no long-run rate effect), but the real effects of the short-term shock in terms of levels of production are permanent: the new growth path is lower than the previous. There is a long-run level effect, with no catch up.

Conclusion

We have considered a bisector economy in which the capitalists aim at maximising accumulation in their own sector. The agents' decisions are based on expectations and are confronted with the constraints resulting from the availability of inputs and with the gap between expectations and achievements. Our model takes money into account, money being considered as a pure means of exchange issued by a bank at the agents' requests (endogenous money). In spite of market clearing, the discrepancies between expected and effective market prices manifest themselves by both real (agents get bundles different from those they had planned) and monetary (monetary imbalances appear) disequilibria. The economy thus moves along a sequence of temporary disequilibria. The study of the real dynamics shows that, in the normal case, the path followed by the relative production converges towards the equilibrium proportion or towards a cycle of order two. The behaviour of that monetary economy differs from that of a barter economy obeying otherwise the same rules (Benetti *et al.* 2012): even if equilibrium is the same in both models, relative prices differ in temporary disequilibria as well as the amounts of goods and their allocations between agents. The same for the dynamics, and therefore money matters.

Why does money, when considered as a pure means of exchange, matter? Because monetary imbalances must be settled at each period. Beyond the arbitrariness of the peculiar rule we have adopted, the basic idea is that the presence of a means of exchange requires the definition of

institutional rules relative to its issue and to balance settlements. Then the allocation of inputs between agents and the dynamics are affected by the existence of monetary imbalances. That analysis of the role of a pure means of exchange contrasts for instance with Keynes's position comparing, in the thirties, a 'real-exchange economy' to a 'monetary economy': according to Keynes, money, as long as it is regarded 'as a convenient means of effecting exchanges', is 'neutral in its effect' because its use does not affect the 'real things' (Keynes, 1933, p. 408).

The definition of institutional rules relative to the issuance of money and to balance settlements completes and modifies the working of a model of a market economy. Ours is an attempt to formalize a bisector economy and may serve as a basis for a macroeconomic study inspired by a Classical approach. It calls for variants concerning legal rules and for extensions: for instance financing monetary balances by transferring securities instead of real capital, or alternatively introducing credit and interest rates.

Appendix 1: Dynamics of the relative quantity

The study concerns the dynamics of the proportion $q = q_1 / q_2$ between produced goods. Formula (12) gives the notional ratio as a function of the previous effective ratio ($q_t^n = f(q_{t-1})$) and formula (13) gives the effective ratio as a function of the notional ratio ($q_t = g(q_t^n)$). The dynamics of the effective ratio q_t are defined by the composite function $h = g \circ f$ given by formula (14). Calculating the derivative shows that f is increasing; as g is increasing if $D > 0$ and decreasing if $D < 0$, the same for function h . Let α denote the dominant eigenvalue of matrix A of technical coefficients and β the other eigenvalue. We have $D = \alpha\beta$, $|\beta| < \alpha$ (according to the Perron-Frobenius theorem), and the sign of D is that of β . By construction, q^* is the stationary solution of (12) and of (13): it is the unique positive fixed point of functions f , g and h . $(q^*, 1)$ being the dominant row-eigenvector of A , we have equalities $a_{11}q^* + a_{21} = \alpha q^*$ and $a_{12}q^* + a_{22} = \alpha$.

Proposition 1 *If $D > 0$, the stationary equilibrium is globally stable.*

Proof If $q_t < q^*$, function $h(x) - x$, which only vanishes at $x = q^*$, is positive for $x < q^*$ and negative for $x > q^*$. As function h is increasing, $q_t < q^*$ implies $q_t < h(q_t) = q_{t+1} < h(q^*) = q^*$ and the sequence q_t converges monotonously towards q^* . The same if the initial position q_0 is such that $q_0 > q^*$. ■

When $D < 0$, as we now assume, function f is increasing but the reaction of the market as described by g is decreasing, so that h itself is decreasing.

Proposition 2 For $D < 0$ and $(1 - \sqrt{2})\alpha < \beta < 0$, there exists a neighbourhood $V =]\bar{q}_{-1}, \bar{q}_1[$ of q^* such that:

- any trajectory starting from V converges towards the equilibrium q^* (local stability);
- any trajectory starting outside V converges towards a cycle of order two.

Proof By using the specific properties of q^* , a simple calculation shows that $g'(q^*) = \frac{D}{\alpha^2} = \frac{\beta}{\alpha}$.

From $f(x) = x^2 / g(x)$ one obtains $f'(q^*) = 2 - \frac{\beta}{\alpha}$. As $h = g \circ f$, we have

$$h'(q^*) = f'(g(q^*))g'(q^*) = f'(q^*)g'(q^*) = -\left(\frac{\beta}{\alpha}\right)^2 + 2\frac{\beta}{\alpha} < 1. \quad \text{The local stability condition}$$

$|h'(q^*)| < 1$ then reduces to $h'(q^*) > -1$, which is met when $(1 - \sqrt{2})\alpha < \beta < 0$. In that case, since h is decreasing, the dynamics oscillate around q^* . Let us consider the function $k = h \circ h$, which

is such that $k(q_t) = q_{t+2}$. Function k is increasing and bounded and may admit fixed points other

than q^* . These fixed points go in pairs: if \bar{q}_1 is a fixed point of k greater than q^* , $\bar{q}_{-1} = h(\bar{q}_1)$ is

another fixed point smaller than q^* . Let $\bar{q}_1 < \bar{q}_2 < \bar{q}_3$, etc. be the fixed points of k greater than

$q^* = \bar{q}_0$. The fixed points smaller than q^* are $\bar{q}_{-3} < \bar{q}_{-2} < \bar{q}_{-1}$ with $\bar{q}_{-n} = h(\bar{q}_n)$. The points \bar{q}_n are

the zeroes of function $k(q) - q$. As $k'(q^*) = [h'(q^*)]^2 < 1$, the function $k(q) - q$ is positive on

$]\bar{q}_{-1}, q^*[$ and negative on $]q^*, \bar{q}_1[$, more generally positive on $]\bar{q}_{2n-1}, \bar{q}_{2n}[$ and negative on

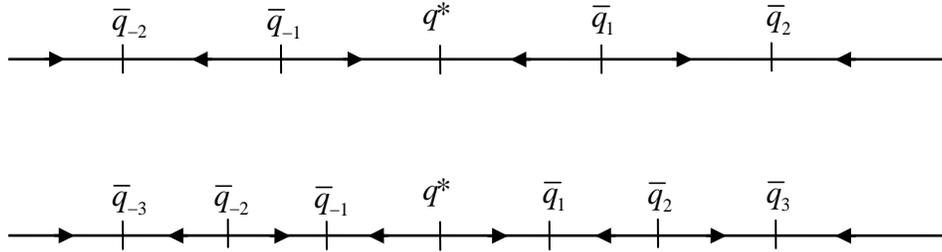
$]\bar{q}_{2n}, \bar{q}_{2n+1}[$.

Function k sends $V =]\bar{q}_{-1}, \bar{q}_1[$ into itself and the sign of $q_{t+2} - q_t = k(q_t) - q_t$ shows that q_{t+2} is closer to q^* than q_t , hence the convergence result on V . For $n \neq 0$, both sets $[\bar{q}_{2n-1}, \bar{q}_{2n}]$ and $[\bar{q}_{2n}, \bar{q}_{2n+1}]$ are preserved by k and the sign of function $k(q) - q$ shows that the successive transforms of a point tend towards \bar{q}_{2n} . Therefore the successive transforms by h of a point belonging to any of these sets tend towards the limit cycle $\{\bar{q}_{-2n}, \bar{q}_{2n}\}$. ■

Proposition 3 *If $D < 0$ and $\beta < (1 - \sqrt{2})\alpha$, any trajectory which does not start from q^* tends towards a cycle of order two.*

(The arguments are similar to those of the previous proof.)

Propositions 2 and 3 are illustrated by the diagrams below.



Appendix 2: The ‘offer-for-sale’ model

The offer for sale model is the monetary version of the real model examined in Benetti *et al.* (2012). The agents’ plans are identical in the ‘offer-for-sale’ model and in the ‘all-for-sale’ model. Formulas (1) and (2) concerning notional quantities, formulas (6) and (7) defining the produced quantities as functions of effective prices as well as relation (9) in the normal case still hold. The main difference lies in the supply and demand functions: $s_i = q_i^- - a_{ii}q_i^n$ and $d_i = a_{ji}q_j^n p_i^e$, hence

$$p_1 = \frac{d_1 p_1^e}{s_1} = \frac{a_{21} q_2^n p_1^e}{q_1^- - a_{11} q_1^n}$$

$$p_2 = \frac{d_2 p_2^e}{s_2} = \frac{a_{12} q_1^n p_2^e}{q_2^- - a_{22} q_2^n}$$

therefore the relative price is

$$p = \frac{q_2^- - a_{22} q_2^n}{q_1^- - a_{11} q_1^n} \frac{a_{21} q_2^n}{a_{12} q_1^n} p^e \quad (18)$$

By inserting (1) and (2) in (18) one obtains

$$p = \frac{p^e}{(q^-)^2} \frac{a_{21}^2 (a_{12} + a_{11} p^e)^2}{a_{12}^2 (a_{22} + a_{21} p^e)^2}$$

The dynamics of the relative price follow from hypothesis (11) on the formation of expectations, from which relation (9) allows us to deduce the dynamics of the relative activity level: we have

$q^+ = h(q)$ or, equivalently, $q_{t+1} = h(q_t)$, with

$$q^+ = h(q) = \frac{1}{q} \frac{(a_{21} + a_{11} q)^2}{(a_{22} + a_{12} q)^2}$$

It comes as no surprise that q^* is the unique fixed point of h . But function h is neither bounded nor necessarily monotonous: calculation shows that the sign of its derivative is that of expression $-a_{11} a_{12} q^2 + (D - 2a_{12} a_{21})q - a_{21} a_{22}$. Therefore, h is decreasing if $D < 0$ but is not necessarily increasing if $D > 0$, so that the dynamics may be very involved. For a local study in a neighbourhood of the equilibrium, one calculates the derivative of function h at q^* :

$$h'(q^*) = q^* \frac{h'(q^*)}{h(q^*)} = q^* (\ln h)' = q^* \left(-\frac{1}{q^*} + 2 \frac{a_{11}}{a_{11} q^* + a_{21}} - 2 \frac{a_{12}}{a_{12} q^* + a_{22}} \right)$$

$$h'(q^*) = -1 + 2q^* \frac{D}{(\alpha q^*) \alpha} = -1 + 2 \frac{\beta}{\alpha}$$

Therefore, the local dynamics are unstable if $\beta < 0$, oscillating and stable if $0 < \beta < \frac{\alpha}{2}$,

monotone and stable if $\beta > \frac{\alpha}{2}$.

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