The Wage-Maximisation Property

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Abstract

For linear single-product models, the competitive long-run technique is wage maximising at a given rate of profit. The property is extended to multiple-product systems that satisfy an additional hypothesis called robustness. In particular, the nonsubstitution property implies that all prices are minimum in terms of wage. The result applies to several types of models, including production with fixed capital. On the robustness hypothesis, the wage-maximisation property can be seen as the counterpart in value terms of the golden rule.

Keywords. Nonsubstitution, wage maximisation, golden rule, fixed capital, Sraffa.

JEL classificaton. B24, C62, D41, D51

1 Introduction

Single-product systems with several available methods of production per industry have two noteworthy long-run properties: nonsubstitution, which is a physical property, and wage maximisation, which is a value property. The first property means that a set of methods which meets the requirements for use represented by some rate of accumulation and some final demand basket can meet any other requirements. This implies that the competitive technique for a given rate of profit is independent of both components of demand. The second property means that, given the rate of profit, the competitive technique maximises the real wage across all techniques. Both results can be established successively and independently. The question we examine

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concerns the existence of a link between these properties: is wage maximisation a consequence of nonsusbstitution? To build such a bridge, we consider a more general framework for which none of these properties holds in general, and show that all systems which have the physical property also have the value property: this, we show, is the case for multiple-product systems. In fact, the nonsubstitution property can be weakened and replaced by two independent conditions linked to the two components of demand.

Section 2 looks at the reason why Sraffa (1960) erroneously extended the wage-maximisation property to multiple-product systems. The main result is proved in Section 3, and Section 4 clarifies its economic interpretation. The next Sections illustrate the strength of the result: in the case of production with capital goods specific to each industry, the properties of the economic system as a whole only depend on those of the subsystem representing the production of capital goods by means of themselves (Section 5). This holds in particular for production with fixed capital, hence a reconstruction and a generalisation of truncation theory (Section 6). Section 7 sketches the history of truncation theory and clarifies the relationship between wage maximisation and the golden rule. Another application of the main result is given in Section 8.

2 Back to Sraffa

The reading of *Production of Commodities by Means of Commodities* is recommended to all economists interested in economic theory. The book is full of stimulating views, and even its mistakes are instructive.

2.1 Sraffa on wage maximisation

Let there be a viable single-product system with homogeneous labour, n methods and n goods. The wage curve represents the the level of the wage as a function of the rate of profit. The curve is decreasing (Ricardian trade-off property), is maximum when the rate of profit is zero (then the wage is equal to the net product per worker) and is zero when the rate of profit reaches its maximum level R. In fact, there are infinitely many wage curves, each depending on the chosen numeraire.

In the presence of two methods 1 or 2 producing the same good, the question of the choice of the operated method arises. It is rather a question of the choice of the operated technique, either the system I or the system II, each made of n operated methods. Each system defines a price-and-wage vector (p_I, w_I) or (p_{II}, w_{II}) which depends on the rate of profit, and the cost-minimising method is chosen because it is more profitable. Sraffa's reasoning in Sections 93 and 94 of *PCMC* follows a thorny path in a deep forest, then leads to a clearing and reaches the following conclusions:

(i) Let the rate of profit r be given. The choice of the cost-minimising technique is consistent since 'the order of the two methods as to cheapness must be the same in the two systems': method 2 yields extra-profits at prices (p_I, w_I) if and only if method 1 pays extra-costs at prices (p_{II}, w_{II}). Therefore, the cost-minimising system is defined with no ambiguity.

(ii) At some exceptional rates of profit, called switchpoints, the two alternative methods are equally costly. Then a unique (up to a scalar) price-and-wage vector (p, w) fits for both systems simultaneously. The wage in terms of any numeraire being the same, the wage curves of systems I and II cross at switchpoints.

(iii) By a thought experiment, let the rate of profit start from its higher possible level R_{II} $(R_{II} > R_I)$ and decrease progressively. For $R_{II} > r > R_I$, only technique II is feasible, and that technique remains cheaper as long as no switchpoint is reached. After a first switchpoint and down to the second, technique I is cost-minimising. By drawing the two wage curves in the same Figure, it turns out that the cheaper technique is always the one on the upper envelope of the wage curves. The same in the presence of several alternative methods in several industries. Hence the wage-maximisation property of the long-run competitive technique, for any given rate of profit. Since the property holds for any numeraire, the competitive prices are minimum when labour is the numeraire $(w = 1)$.

The result is an 'invisible hand effect', i.e. a non intentional effect of competition. It looks more in line with the working of a capitalist economy when it is considered the other way round: the competitive technique is profit maximising for a given real wage. Even from that point of view, the difficulty is to reconcile the ideas of profit maximisation and of minimum price. For instance, Stiglitz (1970) relied on his economic intuition to claim that, for multiple-product systems but when the economic system is oriented towards the production of a unique final good, the competitive price of that good is minimum -but that statement is erroneous.

2.2 Sraffa's mistake

Sraffa presumed that the wage-maximisation property, which he proved for single-product systems, also holds for multiple-product systems, but the justification he gave for that claim is quite different from Stiglitz's. Indeed, Sraffa found in that alleged property a way to solve a specific difficulty he pointed at for multiple-product methods:

> With single-product industries, each process or method of production is identified by the commodity which it produces, so that, when an additional, $(n+1)$th, method is introduced, there is no doubt as to which the pre-existing methods it is an alternative to. When, however, each process or method produces several commodities, and each commodity is produced by several methods, this criterion fails. And the problem arises of how to identify among the pre-existing methods the one to which the new method is an alternative (*PCMC*, Section 96).

The $(n+1)$th method is introduced because it yields extra-profits at the prices associated with the n previous methods, and the problem is to choose the one it replaces. Later in the same Section 96, Sraffa's proposal to solve the selection puzzle is: let the new method be substituted successively for each of the n previous methods, thus leading us to consider n potential systems. A price-and-wage vector is associated with each of them, and the new competitive system is the one which maximises the wage. The answer kills two birds with the same stone: the selection problem is solved and the wage-maximisation property extended to multiple-product systems.

For many reasons, Sraffa's argument cannot be retained. From a logical point of view, Sraffa *proves* the wage-maximisation property for single-product systems and *sets* it for multiple-product systems. Bidard and Klimovsky (2004) have underlined the fragility of the reference to the upper envelope: for some rate of profit, the wages in terms of commodity i may be the same, but not those in terms of commodity j. Therefore, a 'fake switchpoint' appears or disappears according as one or the other good is chosen as numeraire.

2.3 Sraffa on the requirements for use

A contradiction, which does not not seem to have ever been noticed, can be found between the very first and the very last Sections of *PCMC* devoted to joint production. In Section 50, Sraffa argues that, flukes apart, the number of operated processes is equal to the number n of commodities ('square system', in our terminology), and we proceed here as if the argument was fully convincing. Assume $n = 2$ goods to simplify:

> Incidentally, considering that the proportions in which the two commodities are produced by any one method will in general be different from those in which they are required for use, the existence of two methods producing them in different proportions will be necessary for obtaining the required proportion of the two products through an appropriate combination of the two methods (*PCMC*, Section 50, note 2).

'Requirements for use' are the name given to demand (no doubt, Sraffa wanted to avoid any confusion with an approach based on utility theory and leading to a symmetric theory of value). The remark is indeed 'incidental' in *PCMC*, as no further reference to the notion is made in the book -even when Sraffa describes the dynamics of cultivation but only mentions the 'short supply' of land (Section 87) when Ricardo referred to demand explicitly.

In Section 50, Sraffa's argument is that, if goods A and B are required for use in equal proportions, and since the existence of a method producing them in those proportions is highly improbable, the result is achieved by combining a method 1 which produces relatively less of good A with another method 2 which produces relatively more. This suffices to show that the wage-maximisation procedure cannot hold: if some method 3 is introduced because it yields extra-profits, method 3 is substituted for method 1 if it has the same characteristic (it produces less of A), and for method 2 otherwise. In other words, the identification of the outgoing method lies in the physical condition that the requirements for use are met, not in a value condition such as a wage-maximisation property. (The remark sets the grounds of a theory of choice of techniques in a Sraffian framework, developed elsewhere.)

Post-Sraffian scholars have retained the concept of requirements for use and have characterised them by a given rate of accumulation g ($g \leq r < R$) and a given final demand basket d ($d > 0$).

3 A General Property

Let us show that multiple-product systems that satisfy some additional conditions have a wage-maximisation property. These conditions refer to the effects of either to a change in the rate of accumulation (Section 3.1) or in final demand (Section 3.2). They are met if the nonsubstitution property holds, and in that case all prices in terms of wage are minimum for the competitive prices, at a given rate of profit.

3.1 Main result

Consider an economy on a regular growth path at rate g. Let there be n commodities, which are produced by means of themselves and homogeneous labour. The m available methods are represented by a semipositive $m \times n$ matrix A of material inputs, a semipositive $m \times 1$ vector l of labour inputs and a semipositive $m \times n$ matrix B of outputs, obtained one period later (method i is represented in the ith row of these matrices, $i = 1, ..., m$). Constant returns prevail and all commodities can be freely disposed of. The following analysis rests on a precise definifion of a technique:

Definition 1 *Let the requirements for use (g, d) and the rate of profit r be given. A technique is a subset of operated methods which sustain the requirements and is associated with a price-and-wage vector ($p > 0$, $w > 0$) such that all operated methods yield the rate of profit r.*

A competitive technique is a technique for which, moreover, no alternative method yields extra-profits at its associated prices. Formally, a competitive technique is defined by an $m \times 1$ vector y of activity levels and an $(n+1) \times 1$ price-and-wage vector (p, w), solution to the following linear complementarity problem (symbol T denotes transposition):

$$\begin{aligned} d^T &\leq y^T(B-(1+g)A) \quad [p] && (1)\\ Bp &\leq (1+r)Ap + wl \qquad [y] && (2)\\ y &> 0, p > 0, w > 0 && (3) \end{aligned}$$

Inequality (1) means that the physical requirements are met, and that over-produced goods are zero-priced. That inequality may alternatively be written as an equality, provided that the free disposal methods are explicitly written down in (A, l, B): both formalisations will be used in parallel. Inequality (2) means that no method yields extra profits at prices (p, w), and that methods which yield less than the ruling rate of profit are not operated. Only zero-priced goods may be disposed of freely. Sufficient conditions for the existence of a competitive equilibrium are:

(H1) $\left\{y > 0, y^T(B-(1+g)A) \geq 0\right\}$ implies $y^T l > 0$.

(H2) $\exists \overline{y} > 0 \quad \overline{y}^T(B-(1+r)A) >> 0$;

Hypothesis (H1) means that labour is directly or indirectly required for the production of any basket after accumulation at rate g. For simplicity we also retain an indecomposability hypothesis meaning that any good (or any final good in Sections 5 and 6) enters directly or indirectly into the production of all other goods. Hypothesis (H2) means that the rate of profit is smaller than the von Neumann growth rate G_N of (A, B).

In the next two Definitions, the rate of profit r is given and the physical requirements change. Robustness (Definition 2) lets the rate of accumulation g jump up to r, whereas the weak nonsubstitution property (Definition 3) considers a change in basket d.

Definition 2 *A technique sustaining the given physical requirements (g, d) is said to be robust if its subset of operated methods (including possibly some free disposal methods) can also sustain the production of basket d if the rate of accumulation is rised up to r. An economy is robust if all its techniques are robust.*

The price-and-wage vector (p, w) is defined up to a factor and the real wage in terms of basket d amounts to $w/d^T p$. Proposition 1 sets a bridge between robustness and wage maximisation:

Proposition 1 *In a robust economy, the real wage in terms of basket d is maximum for a competitive technique.*

Proof. Let a technique sustaining the requirements (g, d), I its set of operated methods (including free disposal methods, if any) and p its associated price vector ($p > 0$, $w = 1$). By the robustness hypothesis, there exists a vector $z > 0$ of activity levels, with null components outside I, such that $d^T = z^T(B - (1 + r)A)$. Since, at prices p, any operated method i in I yields the rate of profit r, the same holds for any combination of these methods, hence $z^T(B - (1 + r)A)p = z^T l$. Therefore equality $d^T p = z^T l$ holds for any robust technique, with $z^T l > 0$ by (H2). For a competitive technique (its corresponding magnitudes are denoted with a bar), inequality $(B-(1+r)A)\overline{p} \leq l$ also holds. Consider the dual linear programmes (P) and (P*)

$$\begin{array}{llcl} (\mathrm{P}) & \min_{z\geq 0} z^T l \quad \text{s.t. } z^T(B - (1 + r)A) & \geq & d^T \\ (\mathrm{P}^*) & \max_{p\geq 0} d^T p \quad \text{s.t. } \quad (B - (1 + r)A)p & \leq & l \end{array}$$

As $\overline{z}$ is feasible for (P) and $\overline{p}$ for (P*), equality $\overline{z}^T l = d^T \overline{p}$ shows that $\overline{z}^T l$ is the common value $V(P) = V(P^*)$ of these programmes. And since, for any given technique, its associated vector z of activity levels is feasible for (P), we have $d^T p = z^T l \geq V(P) = V(P^*) = d^T \overline{p}$. The real wage $1/d^T p$ is therefore maximum for the competitive technique. ■

When the rate of profit increases, the constraints in (P*) loosen, therefore $V(P^*)$ increases and the real wage $\overline{w} = 1/d^T \overline{p} = 1/V(P^*)$ decreases (Ricardian trade-off property).

3.2 Weak nonsubstitution

The notion of robustness is complemented by that of weak nonsubstitution:

Definition 3 *(Weak Nonsubstitution). For a given rate of accumulation, a technique has the weak nonsubstitution property if it can adapt its activity levels to meet any change in final demand. An economy has the weak nonsubstitution property if all its techniques have it.*

If an economy is robust and has the weak nonsubstitution property, the wage-maximisation property holds for any basket, therefore all components of the competitive price vector are minimum ($w = 1$) for a given r and, in particular, the competitive price vector is unique. To sum up, checking properties of the physical side suffices to prove important properties of the value side.

A technique defined by the requirements (g, d) and the rate of profit r is said to have the nonsubstitution property if it can meet any other requirements (g', d') with $g' \leq r$. An economy has the nonsubstitution property if all its techniques have it. This, it is well known, is the case for a basic single-product economy. Formally, the property follows from the Perron-Frobenius properties: equality $(1 + r)Ap + l = p$ for some semipositive price vector implies that the inverse matrix $L_{g'} = (I - (1 + g')A)^{-1}$ is positive for any $g' \leq r$, and therefore arbitrarily given requirements (g', d') are sustained by activity levels $y' = L_{g'}^T d'$. There follows that a basic single-product economy has the wage-maximisation property.[1]

[1]Note however that Proposition 1 assumes free disposal. For a basic single-product economy, one may first admit free disposal as a temporary hypothesis, then show that any price vector sustaining a technique is positive, so that no free disposal method is operated, and conclude that the temporary hypothesis may be dispensed with.

4 Min or max?

The proof of Proposition 1 requires a conceptual clarification. Dual programmes are considered: in economic terms, programme (P) aims at defining an efficient allocation of labour among the available methods, for given requirements (r, d); by contrast, as noticed by Schefold (1980), the economic rationale of (P*) is less straightformard, since its objective is to *minimise* the real wage $1/d^T p$ in terms of basket d. Schefold's interpretation refers to von Neumann (1945-6), who defines a rate of interest r for a given price vector as a scalar for which no method yields more than r. The set of interest rates is then characterised by its lower bound R_N whereas, in the classical tradition, the set of the rates of profit admits an upper bound: the two concepts are distinct. Von Neumann's main result is that the minimum interest rate R_N coincides with the maximum growth rate G_N of the economy. Even if von Neumann's model does not consider labour explicitly, the logic of its construction, Schefold argues, fits with programme (P*) and the search for a minimum wage.

An answer more in line with Sraffa's approach relies on a simple but apparently unnoticed property of linear programmes. The duality theory associates linear programmes by pairs (P) and (P*) which have the same value. We show the existence of two more 'joint' programmes (JP) and (JP*) which also have the same value, and (JP*) admits a simple economic interpretation.

Let $(\overline{z}, \overline{p})$ be a solution of the linear programmes (P) and (P*)

$$\begin{array}{lllll} \text{(P)} & \min\limits_{z \geq 0} z^T l & \text{s.t. } z^T C & \geq & d^T \\ \text{(P}^*\text{)} & \max\limits_{p \geq 0} d^T p & \text{s.t. } \;\; Cp & \leq & l \end{array}$$

with common value $V(P) = V(P^*)$ and consider the joint (nonlinear) programme (JP*):

$$\text{(JP}^*\text{)} \qquad \min_{p \geq 0, y \geq 0} d^T p \;\; \text{s.t.}$$

$$z^T C \;\; \geq \;\; d^T \qquad [p] \tag{4}$$

$$z_i \;\; > \;\; 0 \Rightarrow (Cp - l)_i = 0 \tag{5}$$

On the one hand, since the pair $(\overline{z}, \overline{p})$ satisfies the constraints (4) and (5), the value of the minimisation programme (JP*) is at most equal to $d^T \overline{p}$, therefore $V(JP^*) \leq V(P^*)$. On the other hand, for an arbitrary pair (z, p) satisfying the constraints (4) and (5), we have $d^T p = z^T Cp = z^T l$. As z is feasible for the minimisation programme (P), inequality $z^T l \geq V(P)$ holds, therefore $d^T p \geq V(P)$. Since that inequality holds for any feasible pair (z, p) of (JP*), we have $V(JP^*) \geq V(P) = V(P^*)$. The overall conclusions are that $V(JP^*) = V(P^*)$ and that $(\overline{z}, \overline{p})$ is a solution of programme (JP*). [2] [3] The noteworthy fact is that a direct comparison of programmes (P*) and (JP*) shows that the objective is switched from maximisation to minimisation.

For $C = B - (1 + r)A$, programme (JP*) considers all techniques which, on the physical side, can meet the requirements (r, d) and, on the value side, are such that all operated methods yield the rate of profit r. Within that set of techniques, the competitive technique is wage-maximising in terms of basket d. The robustness hypothesis extends the result to the rate g of accumulation, instead of r.

5 Production with internal capital goods

Production with fixed capital ('machines') is a type of multiple-product system, as pointed out by Torrens: if production in industry i involves one machine, the yearly product of a process making use of a machine of age t is made of a certain amount of final good i and a machine of age $t + 1$. However, when the machine produces the same final good all along its life (non-transferability hypothesis), its overall intertemporal production looks like that of a single-product industry. It is therefore expected that the economic behaviour of a productive system with fixed capital is basically the same as that of a single-product system. Truncation theory justifies that intutition, provided that the economic life of the machine may be shortened with regard to its physical life (this is but the free disposal axiom applied to machines) and is determined endogenously as a function of distribution. Applying the above results to production with fixed capital reduces truncation theory to a few lines: it suffices to show that a standard fixed capital model has the nonsubstitution property.

The path we follow is somewhat slower, as it is decomposed in steps, with

[2] The argument still holds when equality $(CP - l)_i = 0$ in implication (5) is replaced by inequality $(CP - l)_i \geq 0$.

[3] The writing down of programme (JP) and the proof of equality $V(JP) = V(P)$ are left to the reader.

the idea to attach each property with a specific characteristic of production and to extend the results to models of a more general type. In the present Section, we ignore the intertemporal nature of production with fixed capital and only retain that the commodities called 'machines' are capital goods proper, which are used in the production of a unique final good. Proposition 2 shows that the productive system has the weak nonsubstitution property, and Proposition 3 that an additional condition relative to the structure of production of these capital goods by means of themselves ensures the nonsubstitution property. This once done, the truncation theory will be reduced to an even more elementary property: checking that the additional condition is met in the case of production with fixed capital.

Production with internal capital goods relies on the existence of a partition of the n commodities into f final goods, which are consumed and may also be used as inputs, and $n-f$ capital goods proper, which are not consumed. The set of capital goods proper is itself partitioned into f subsets, each attached to a final good, and it is assumed that condition (C1) holds:

(C1) Each method produces one final good at most and, possibly, a number of capital goods attached to it: any of these capital goods is involved (i.e. used or produced) into the production of a unique final good.

Let industry i ($i = 1, ..., f$) be the set of methods producing the final good i. Condition (C1) means that each capital good is internal to an industry and not transferable to another: the material inputs of a method belonging to the ith industry consist of final goods and capital goods attached to i, and its outputs consist of good i and its attached capital goods. Proposition 2 looks at the adaptation of the productive system to a change in final demand and shows that the weak nonsubstitution property holds.

Proposition 2 *Let g be given. On condition (C1), a competitive technique at the rate of profit r can meet any final demand basket for final goods.*

Proof. Consider a competitive technique sustaining the production of some semipositive final demand basket at growth rate g ($g \leq r$), and let y_i be the semipositive vector of activity levels in industry i. By applying activity levels y_i, the whole industry i is reduced to a unique representative process $(a_i, c_i, l_i) \to (b_i, (1+g)c_i)$, where a_i is a vector of final goods, c_i a vector of capital goods proper, and b_i a scalar representing an amount of final good i. Let p be the price vector of final goods and π that of capital goods ($w = 1$). Equality

$$(1+r)(a_i^T p + c_i^T \pi) + l_i = b_i p_i + (1+g)c_i^T \pi \tag{6}$$

holds and implies that

$$(1+r)a_i^T p + l_i \leq b_i p_i \tag{7}$$

Consider the fictitious single-product system made of the f processes $(a_i, l_i) \to b_i$. By inequality (7), its maximum rate of growth is greater than r, therefore any arbitrarily given basket $d \in R_+^f$ can be obtained as a g-net product by adjusting the activity levels. For the original productive system, this means that, when the vectors y_i of activity levels are replaced by $\lambda_i y_i$, where the positive scalars λ_is are adequately chosen, the amount of final goods after accumulation at rate g is equal to d. Since the operation leaves the relative activity levels inside each industry unchanged, the g-net product of the internal capital goods remains nil. ■

The notion of robustness invites us to contemplate the effect of a rise in the rate of accumulation from g to r. Then the feature which matters is the production of capital goods by means of themselves. In matrices (A, B), let us therefore isolate the columns (C, D) corresponding to capital goods proper. Non-transferability means that they are made of f independent sub-matrices (C_i, D_i) with different capital goods. For a technique sustaining the requirements (g, d), the activity levels y_i within industry i are such that, when the free disposal methods are explicitly written down, equality $y_i^T D_i = (1+g) y_i^T C_i$ holds. The following axiom (C2) restricts the robustness hypothesis to the internal capital goods:

(C2) Isolate the capital goods of industry i. Any subset of operated methods which sustains the growth rate g can also sustain the growth rate r.

In formal terms, axiom (C2) is written:

$$\forall i \quad \left\{ \exists y_i > 0 \;\; y_i^T D_i = (1+g) y_i^T C_i \right\} \Rightarrow \left\{ \exists z_i > 0 \;\; z_i^T D_i = (1+r) z_i^T C_i \right\} \tag{8}$$

with $\mathrm{supp}(z_i) \subset \mathrm{supp}(y_i)$. The next proof shows that this condition suffices for the robustness property to hold for final goods also, therefore:

Proposition 3 *In an economy satisfying conditions (C1) and (C2), the competitive prices of the final goods in terms of wage are positive, unique and minimum.*

Proof. For a given rate of profit, let y be the activity levels of a technique sustaining the requirements (g, d), and y_i be the sub-vector corresponding to industry i. By axiom (C2), y_i can be changed into other activity levels z_i involving the same operated methods, and such that the capital goods are accumulated at rate r. The proof of Proposition 2 can be repeated after replacement of y_i by z_i: by applying activity levels z_i, the whole industry i is reduced to a unique representative process $(a_i, c_i, l_i) \to (b_i, (1+r)c_i)$. Equality $(1+r)a_i^T p + l_i = b_i p_i$ holds, therefore there exist activity levels $\lambda_i z_i$ sustaining the r-net production of an arbitrary final demand basket, and after that change the r-net product of capital goods remains nil. To sum up, any technique is robust. Since, however, the final demand basket is of the type $(d, 0)$ the price-minimisation property holds for the final goods only. ■

As a first application of these results, consider an economy with at most one capital good proper in each industry and suppose that, for any method, the self rate of reproduction of that good does not lie in the interval $[g, r]$. Then any technique sustaining the growth rate g must combine at least two operated methods in each industry, one with a self rate greater than g (and therefore than r), the other smaller than g (possibly but not necessarily, a free disposal method). Since these methods can sustain the rate of accumulation r for capital goods, axiom (C2) is met and the competitive prices of final goods are minimum.

6 Production with fixed capital

Axiom (C2) also applies to production with fixed capital. We start from the standard formalisation of fixed capital and next extend it.

The structure of production with fixed capital is: there are f final goods ($i = 1, ..., f$) determining f industries, with at most one type of machine per industry. The machines are capital goods proper. Each type of machine is involved in a unique industry, and its different varieties (qualities) are identified by an age index $t = 0, 1, ..., T-1$. In industry i, a new machine M_0 is produced by means of final goods only and, up to age T, one machine M_t of age t is produced by means of one machine of age $t-1$. T is the physical lifetime of the machine: a machine of age T has no scrap value and may be ignored. Formally, the corresponding $(T+1) \times T$ input matrix C_i has a zero first row and is filled with zeroes except for a subdiagonal made of ones; symmetrically, the output matrix D_i is the identity matrix completed by a last row made of zeroes:

$$C_i = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(That simplified representation ignores the possibility, taken into account in the formalisation, of alternative uses within industry i of a machine of any age: then, the same row is repeated several times in (C_i, D_i).) Machines apart, the inputs of the yearly processes are made of final goods and labour, and final good i is the only one produced in industry i.

Proposition 4 *An economy with fixed capital has the nonsubstitution property for final goods. The competitive prices of final goods depend on the rate of profit only, are positive, unique and minimum in terms of wage. The trade-off property holds.*

Proof. At a competitive equilibrium for some given requirements (g, d), let the economic lifetime of the machine in industry i be reduced to τ (truncation), the machine of age τ being disposed of freely. In that industry (index i is ignored), the number α_t of machines of age t amounts to $\alpha_0(1+g)^{-t}$ for $t < \tau$, and to 0 for $t \geq \tau$. Replacing the activity level y_t of any method using a machine of age t by $z_t = y_t(1+r)^{-t}(1+g)^t$ shows that condition (C2) is met. Proposition 3 applies. ■

The standard formalisation of fixed capital is too restrictive, however, in the following case. Imagine that a new machine can be used either under normal or intensive conditions, and that the machine obtained after one year of intensive use is of the same quality as after two years of normal use. Each variety of the machine can no longer be identified by an age index. However, these varieties have still a hierarchical character and one can attribute to each of them a quality index q, such that a machine of quality q is produced by a machine of a smaller quality index. Even more generally, the hierarchical structure still holds when, in any industry, a capital good of quality q is produced by one or several capital goods of lower qualities: then matrix C_i is subtriangular (semipositive entries below the ones in C_i). Axiom (C2) is still met and, with no further calculation, it can be said that the economic behaviour of that type of model is the same as that of a single-product economy.

7 Historical background

The idea of truncation was first elaborated in a neo-Austrian framework: a neo-Austrian model is a one-good model in which an intertemporal flow of labour or, for a given real wage, of the good generates an intertemporal flow of outputs. Its associated rate of interest should be the internal rate of return, but multiple rates may coexist (Fisher, 1907). The idea of shortening the intertemporal process by stopping the investments after a certain date is quite natural in the neo-Austrian approach, which stresses that the 'average period of production' is sensitive to distribution. The truncation theory was elaborated at the end of the fifties (Soper, 1959; Karmel, 1959; Silcock, 1959; Wright, 1959; results independently rediscovered by Arrow and Levhari, 1969). Its main conclusion is that the competitive truncation corresponds to the maximum maximorum internal rates of return across all truncations. For that truncation, the internal rate of return is indeed unique. However, in general, there is no negative correlation between the rate of interest and the duration of the process, contrary to Austrian ideas.

For multisector models, the theory of truncation started with Morishima (1963) and was completed by Schefold (1971, 1978b), Baldone (1974) and Varri (1974) in a Sraffian framework, the one we have adopted. Mirrlees (1969) pointed at the different roles played by final demand and accumulation and introduced the idea of weak nonsubstitution (our terminology), which was resumed by Stiglitz (1970). The same distinction was clarified in a Sraffian framework by Salvadori (1988). The papers on truncation theory make use of off-putting calculations and substitute an artificial partition of commodities between final goods and new machines on the one hand ('finished goods') and old machines on the other hand for the natural partition between final goods and machines (Bidard, 1997).

The present paper improves upon Bidard (2004) which did not introduce the notion of robustness for multiple-product systems and axiom (C2) for capital goods proper. The wage-maximisation property is intimately linked with the physical possibility to jump from the actual rate of accumulation g up to the rate of profit r. Salvadori's paper (1982) may be reinterpreted in those terms. We have already mentioned Schefold (1980) who wrote down the dual programmes (P) and (P*) for $g = r$ and was puzzled by an apparent wage-minimisation property.

In a one-good neoclassical model with a smooth production function, the golden rule $g = r$ is endowed with a distinctive property: the net product per worker after accumulaton is then maximum (Allais, 1947; Desrousseaux, 1961; Phelps, 1961). Under the same assumption, the net product coincides with the wage. A neoclassical production function is robust, but multisector models are the adequate framework to study the wage-maximisation property. In those models, the robustness hypothesis has two consequences: (i) the optimal allocation of labour across the available methods, in order to maximise the net product per worker, is qualitatively the same if the rate of accumulation is either at its effective level g or at the potential level r; (ii) the competitive real wage at a given rate r of profit is the same in both cases. Then the bridge between the maximisation of the g-net product and that of the real wage takes the form of the equivalence between the programmes (P) (optimal allocation of labour) and (JP*) (wage maximisation) examined in Section 4: for multisector models, wage maximisation is the 'joint dual' property in the value space of a golden rule property in the physical space.

8 Factorised economies

This Section provides another application of Proposition 1. The multiple-product systems here considered retain a particular feature of basic single-product economies: In a factorised economy for some given g, it is assumed that the set of methods is partitioned into n subsets S_1, ..., S_n such that the production of any semipositive g-net basket requires at least one method from each subset. The set of techniques is therefore included into $S_1\times \ldots \times S_n$. For basic single-product systems, S_i is the set of methods producing good i (industry i). For multiple-product systems, the notion applies when each method has a dominant product and also in less obvious cases.

Suppose first that there is no choice of methods, each sector S_i being reduced to a unique method. With n goods and n methods, the factorisation hypothesis is written:

(F) The set $Y_g = \left\{y; y > 0, y^T(B-(1+g)A) > 0\right\}$ is nonempty and included into $R^n_{++} = \{y; y >> 0\}$.

Since all methods must be operated, the economy is said to be g-all-engaging (Schefold, 1978a; Bidard, 1996).

Lemma 1 *A g-all-engaging technique has the weak substitution property and is robust.*

Proof. Let $y_0 \in Y_g, y_0 >> 0$, be given. If there existed a nonzero vector y such that $y^T(B-(1+g)A) = 0$, some linear combination of vectors y_0 and y would be semipositive with a zero component and would sustain the production of a semipositive g-net product: a contradiction with assumption (F) would be obtained. Therefore matrix $L_g = (B-(1+g)A)^{-1}$ exists. Consider the set $J = \left\{y; \exists d > 0 \;\; y^T = d^T L_g\right\}$. J contains y_0 and is convex. If some point y in J were outside R^n_{++}, some convex combination of y_0 and y would be on the frontier of R^n_{++} and again a contradiction with assumption (F) would be obtained. Therefore, J is included into R^n_{++} or, in other words, implication $\left\{d > 0 \Rightarrow d^T L_g >> 0\right\}$ holds: this shows that matrix L_g is positive. The g-net production of any given basket $d > 0$ being sustained by the positive activity levels $y^T = d^T L_g$, the weak nonsubstitution property holds.

Let r be such that $g \leq r$ and all methods yield the rate of profit r at some price vector $p > 0$. The set denoted $Y_r^T = \left\{p; p > 0, p^T(B^T - (1+r)A^T) > 0\right\}$ is nonempty. Since $A \geq 0$, we have $Y_r^T \subset Y_g^T$ and, by the positivity of $L_g^T, \emptyset \neq Y_r^T \subset Y_g^T \subset R^n_{++}$. This shows that condition (F) is met by the matrix $B^T - (1+r)A^T$. By the same argument as above, the inverse matrix L_r^T exists and is positive. The r-net production of basket $d > 0$ is sustained by the activity levels $y = L_r^T d$, hence the robustness property. ■

The positivity of matrix L_r shows that prices $p = wL_r l$ are positive. Hence:

Proposition 5 *In a factorised economy and at a given rate of profit, the competitive prices in terms of wage are unique, positive and minimum.*

A significant gap with single-product systems is that the factorisation property is conditional to a level of g and may be lost for some lower rate of accumulation. Therefore, a factorised economy for a certain g provides an example of an economy which has the robustness property and the weak nonsubstitution property but may not have the nonsubstitution property: the conditions considered in Definitions 2 and 3 are less requiring than nonsubstitution.

9 Conclusion

For a productive system on a long-run path, wage maximisation is a value property related to the physical properties of the system and, more precisely, to the possibility for a technique to sustain a hypothetical rise of its present rate of accumulation up to the rate of profit. Checking the robustness property provides a general guide for the study of multiple-product systems.

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