Dynamic platform competition 'In' and 'For' the market*

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Abstract

In many industries, platforms compete with incumbents that are open to all consumers, whereas platforms require user affiliation. Consequently, platforms face two layers of competition: *for the market*, to attract users, and *in the market*, to compete with incumbents. We develop a dynamic model integrating these layers, showing that as platform affiliation grows, in the market competition intensifies, pushing incumbents toward more aggressive pricing. Conversely, for the market competition diminishes, reducing the platform's incentive to compete aggressively. This interplay generates dynamic pricing behavior that can be non-monotonic over time, capturing the shifting incentives driving platform-incumbent competition across both dimensions.

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1 Introduction

Digital platforms have entered many sectors of the economy, challenging and disrupting incumbents, a typical example being ride-sharing platforms (Uber, Lyft, Bolt). These platforms entered a traditional sector (taxi rides) with a new business model. They offer services to customers that are comparable to those of traditional taxi companies, but those platforms have no vehicle fleet. Taking advantage of digital technologies, the company operates as a matchmaker between consumers and drivers while traditional taxi services are organized as vertically integrated service suppliers. Platforms bring to the market many innovations, flexible supply, efficiency gains and they have gained large market shares (Cramer and Krueger, 2016).

But, despite their larger user base and a spectacular growth in the number of users and rides, these platforms struggle to make a profit and have operated at losses for long.[1] The traditional explanation for these losses is that platforms are competing for the market. At the early stages, they need to establish a large user base and they will, eventually, make money later. Competition for the market implies that platforms may sell services at losses to build their user base. Operating losses at early stages are funded by investors, who expect to recover their investment later, when the platform manages to establish a strong market position. Competition is akin to a winner takes all contest. And, we observe a similar competition between incumbents and platforms in many other sectors: short-term rental, food delivery, freight and logistics, digital payment solutions, etc.

The objective of this paper is to model the competitive process between a platform and a vertically integrated incumbent. One fundamental point is that the platform and the incumbent are in an asymmetric position vis-a-vis the consumers. While anyone can buy from an incumbent, to buy from the platform, a consumer must be affiliated. This requires downloading the application, creating an account and sharing some data with the platform. While this is not necessarily costly[2], all consumers are not affiliated to the platform and, usually, affiliation grows over time.[3]

There are several reasons for not to be affiliated to a platform. The first and the most obvious is that a consumer may lack of information about the service and its existence. But

[1]Uber turned a profit in 2023, while Lyft is currently making losses. For many years, these companies had negative operational income, implying that rides were sold at a loss.

[2]Except for using mobile internet, which might be costly especially abroad.

[3]For instance, the number of booked nights on Airbnb grows from 185.5 millions in 2017 to 492 millions in 2024 (source Statista `https://www.statista.com/statistics/1193532/airbnb-nights-experiences-booked-worldwide/`).

on top of that, a consumer may lack of trust or confidence on the service or the service and the app might be complex to use, and there are many other obstacles to adoption.[4] For consumers, it may take time to overcome these barriers and become a user of the new service.

For a platform, it is critical to build a large base of users. And it can use many to attract consumers and boost affiliation. First, it can enhance service quality by adding new and innovative functionalities or by increasing affiliation on the supply side of the platform, leveraging on cross-group network externalities. The platform should also build trust and confidence among users by investing in reliable reputation and feedback mechanisms. Second, consumers need to be aware of the platform. This can be achieved through advertising or word-of-mouth communication, such as on social networks, to spread information about its services. Lastly, the platform may charge low and attractive prices to boost affiliation. And, it is indeed common for a platform to offer discounts, coupons, and sponsored affiliation programs to attract new users and retain early adopters. In this paper, we will assume that by charging lower prices, the platform can convince new users to affiliate with it.

Hence, consumers can be divided into two groups: affiliated consumers, who can buy from both the platform and the incumbent, and non-affiliated or captive consumers, who can only buy from the incumbent. Note that affiliation to the platform merely provides an additional choice option for consumers. Given this, the platform competes on two levels: it competes *for the market* to affiliate new users to its service, and it competes *in the market* with the incumbent for service provision to the affiliated customers; non-affiliated users being served exclusively by incumbents. We construct a dynamic model that integrates and distinguishes these two levels of competition.

The intensity of competition in the market depends on the platform's penetration. With a small base of affiliated users, the competitive threat to the incumbent is limited. However, when the platform has a large user base, this threat becomes more intense. Consequently, the incumbent will be more aggressive in the price setting game when facing a platform with more affiliated users (see Valletti et al., 2002 and Gautier and Wauthy, 2010 for a similar analysis). Competition in the market implies that as the share of affiliated users grows, competition intensifies and prices decline. The incumbent, being more exposed to competition, experiences lower profits, while for the platform, prices decline but market coverage increases. The combination of these two effects makes the platform's profit concave with affiliation.

To integrate competition for the market, we use a dynamic model where affiliation to the platform changes over time. In our model, the affiliation process is driven by words of mouth.

[4]Several papers have studied the determinants of participation in the platform economy (Min et al., 2018; Lee *et al.*, 2018).

Potential users learn over time about the existence of the service and decide to affiliate or not to the service. We will suppose that this affiliation process is accelerated if the platform charges a lower price for its services, particularly at the early stages of growth. Consequently, the platform has incentives to charge a price below the static optimal price, thereby reducing current profits to increase affiliation to its service and achieve larger profits in future periods. In our model, we identify a competition for the market effect, which makes the platform more aggressive in the price-setting game. However, the platform's incentives to lower its price decrease as affiliation increases i.e. the competition for the market effect fades out as affiliation grows.

Our model identifies and distinguishes a competition for the market effect and a competition in the market effect and these two effects are going in opposite direction. The competition in the market effect intensifies competition as platform grows. The competition for the market effect fades out as platform grows. As a result, the platform may experience a non-monotonic price path. At its early stages, prices are low as affiliation of new consumers is critical. As affiliation grows, competition for the market becomes less intense, leading the platform to increase its price. However, as it grows, competition in the market intensifies and, to compete with the incumbent the platform ultimately decreases its price.

We extend our baseline setting in two directions. First, we consider the case where the market tips and the incumbent quits the market when the platform reaches a critical size. We show that market tipping intensifies competition for the market at early stages. Second, we consider that the platform is two-sided and needs to affiliate both consumers and service providers. Two-sideness makes the consumer and producer prices interdependent. Affiliation on the two sides reinforces each other and makes competition for the market more intense. But it also raises the platform's cost which countervail the effect of affiliation.

The paper is organized as follow. We review the literature in Section 2. We present the main elements of our model in Section 3. In Section 4, we identify the equilibrium prices and profits in the static game, that is for a given affiliation to the platform. We introduce the dynamic of affiliation in Section 5. In Section 6, we consider the dynamic game where affiliation changes (endogenously) over time, first with myopic firms, then with forward-looking firms. In Section 7, we present two extensions of the model and Section 8 concludes. The proofs of our main results are in Appendix A . Appendix B illustrates our main results with the Singh and Vives model with linear-quadratic preferences.

2 Related literature

The economic literature has distinguished between competition in and for a market for long. Competition for a market is often associated with markets exhibiting strong economies of scale both on the demand and the supply side (Geroski, 2003). In such circumstances, the market is likely to be dominated by a single firm, with firms competing to win the market. This type of competition is often represented by a battle between different incompatible standards. In this contest, firms use various price and non-price strategies to attract consumers (Besen and Farrell, 1994). Halaburda et al. (2020) integrates network effects and consider a model of competition for the market with platforms of different qualities. They show that, history matters and the market does not always select the higher quality platform.

If firms standardize and make their products compatible, the nature of competition changes. Firms are no longer competing for the market but are competing in the market. Katz and Shapiro (1985) is a seminal paper that analyzes the incentives of firms to offer standardized goods, i.e., the choice between competition in and for the market. Competition in the market has been analyzed in many different contexts, often in static models. Several papers have focused on competition between different business models (Hagiu and Wright, 2015a, 2015b). Our model is a dynamic model that integrates these two dimensions: competition in and for the market.

In many circumstances, the price of product varies during the product life-cycle and firms adopt dynamic pricing strategies. One of the strategy that is often used is *penetration pricing* that consists in offering, at the early stages, new products at a price below their market price.[5] Spann et al. (2015) show that such a strategy is effectively used but only by a fraction of the market participants.[6] More generally, when demand is a function of past or cumulative sales, for instance because demand exhibits network or word-of-mouth effects, the optimal price path may be such that prices are, at early stages, below their profit maximizing level (Kalish, 1983).

Several papers have focused on dynamic pricing of network goods. Cabral et al. (1999) describes the equilibrium price path when a monopolist sells a durable good exhibiting network effects. He derives the conditions for having an increasing price path and he shows that network externalities are critical to have a low introductory pricing in equilibrium. Cabral (2011) extends the model to competing network goods and Cabral (2020) to two-sided plat-

[5]Penetration pricing is different from pricing on two-sided platforms where platforms optimally set low, or even zero price on one side of the platform permanently (Caillaud and Jullien, 2003).

[6]In their study *skimming pricing*, that is a price above the market price, is as often used as penetration pricing.

forms. Shin (2017) considers an efficient monopolistic pricing for a social network to which consumers subscribe (or not) at every period. The monopolist faces a trade-off between increasing future profits by encouraging more consumers to buy today and maximizing current profits, as in our paper, but affiliation is not cumulative and the optimal prices oscillate around their steady state level. Zhu *et al.* (2025) have a model of two periods between platforms. They show that price competition is fiercer in the first period to establish a large user base and benefit from a larger affiliation in the second period.

Farboodi et al. (2019) consider a dynamic model of competition where firms can use accumulated data to better forecast a key parameter of the demand function. Data is a by-product of production and, at each point of time, firms decide on their production level. They show that firms may produce more than the optimal static output in order to collect more data in the initial phase and become more profitable later, a mechanism that has similarities with our model of competition in and for the market.

Several papers have studied empirically the impact of the entry of platforms on the pricing and the performance of the incumbent players. Many papers focus on the impact of Airbnb's entry on hotel revenue, prices, and occupancy rates. Zervas et al. (2017) use data from the hotel industry in Texas and show that an increase in the number of Airbnb listings reduces the revenue per room and they find that hotels respond to Airbnb's entry by decreasing the room prices. Farronato and Fradkin (2022) use a larger dataset covering 50 US cities and confirm these findings. They show that the lower revenue per hotel room is mainly driven by a reduction in room prices, rather than a lower occupancy rate. Chang and Sokol (2020), using data from Taiwan, show that lower-quality hotels react to Airbnb's entry by decreasing their room prices, while higher-quality hotels increase their service quality. Seamans and Zhu (2014) study the impact of the entry of Craiglist, a platform providing classified-ad services, on newspapers. They provide similar evidence that newspapers decrease the classified-ad rate after entry.

3 The model

In this paper, we construct a dynamic model of competition between vertically integrated incumbents and a platform. The main feature of this model is that, while anyone can take a taxi, to be a client of the platform, a consumer must be affiliated to the platform, that is downloading the app, creating an account, agreeing on the terms of services etc. As a consequence, the set of potential clients differ for the incumbents and the platform, but it will evolve over time.

3.1 Firms

We consider a vertically integrated incumbent[7] (indexed by I) and a platform (indexed by P) providing differentiated goods or services to consumers and competing in prices for consumers.[8]

The incumbent and the platform have a different business model. The incumbent is vertically integrated and manages all the supply chain, while the platform is a matchmaker between independent service providers and the clients. We will not enter into the details of the different production processes and treat them as blackboxes. We suppose that firms have a marginal cost c^i and fixed cost f^i, $i = I, P$.

3.2 Consumers

There is a population of consumers of size one. We will use index j to refer to an arbitrary consumer. There are two types of consumers, those who are affiliated to the platform (in proportion α) and those who are not (in proportion $1 - \alpha$).

An affiliated consumer can buy from the incumbent and/or from the platform; a non-affiliated consumer can only buy from the incumbent. Affiliation to the platform only offers an *additional choice* to the consumers who has, of course, no obligation to buy from P.[9] In our model, we will assume that, beside being affiliated or not, all consumers are symmetric.

3.3 Demand functions

Given the prices (p^I, p^P) charged by I and P, we define the individual demand functions for each product. We distinguish the demand from affiliated and non affiliated consumers.

If a consumer j is affiliated to the platform, his demand for the services of the incumbent

[7] In our model, we can conduct similarly the analysis if we consider, instead of a single incumbent, a fringe of N incumbents (see footnote 11). The single incumbent assumption should be viewed as a simplifying shortcut for the more realistic fringe situation.

[8] Some platforms, like ride-sharing platforms, also compete on the other side of the market for service providers, in this case taxi drivers. We consider an explicit two-sided platform as an extension of the baseline model.

[9] Guttentag and Smith (2017) show, based on survey data, that Airbnb and hotel are substitute for a large proportion of Airbnb guests.

I and the platform P are denoted by:

$$\tilde{x}_j^I(p^I, p^P) \text{ and } \tilde{x}_j^P(p^I, p^P) \text{ if } j \text{ is affiliated.}$$

If a consumer j is not affiliated, his demand for the platform is trivially zero, and the demand for the incumbent I is:

$$x_j^I(p^I) \text{ and } x_j^P = 0 \text{ if } j \text{ is non-affiliated.}$$

We make the following assumptions on the demand functions:

Assumption 1 *1. Law of demand:* $\frac{\partial \tilde{x}_j^I}{\partial p^I} < 0, \frac{\partial x_j^I}{\partial p^I} < 0$ *and* $\frac{\partial \tilde{x}_j^P}{\partial p^P} < 0$*.*

2. Product substitutability: For $k, l = I, P$ *and* $k \neq l$*:* $\frac{\partial \tilde{x}_j^k}{\partial p^l} > 0$*.*

3. For given prices (p^I, p^P)*:* $x_j^I(p^I) \geq \tilde{x}_j^I(p^I, p^P)$*.*

4. $\frac{\partial \tilde{x}_j^I(p^I, p^P)}{\partial p^I} \leq \frac{\partial x_j^I(p^I)}{\partial p^I} \leq 0$*.*

Part (1) is the law of demand. In part (2), we assume that the products of I and P are substitutes. In part (3), we suppose that a non-affiliated consumer has a (weakly) higher demand for the incumbent's product than an affiliated consumer. In part (4), we assume that a change in price has higher impact on the demand of I from a non-affiliated consumer than from an affiliated one. In the Appendix B, we use the Singh and Vives (1984) quadratic utility model to illustrate our model.

3.4 Profits

Using the above demand functions, we can define the firms' profit functions. To simplify reading, we drop the j subscript for individual demands, as being affiliated or not, consumers are identical.

First, let us a define a generic profit per user. The incumbent's profit from an affiliated and a non-affiliated user are respectively equal to:

$$\begin{aligned} \tilde{\pi}^I(p^I, p^P) &= (p^I - c^I)\tilde{x}^I(p^I, p^P), \\ \pi^I(p^I) &= (p^I - c^I)x^I(p^I). \end{aligned}$$

Hence, the total profit of an incumbent i is equal to the sum of the profit from the α affiliated consumers and the $1-\alpha$ non affiliated one, net of the fixed cost:

$$\Pi^I(\alpha, p^I, p^P) = \alpha\tilde{\pi}^I(p^I, p^P) + (1-\alpha)\,\pi^I(p^I) - f^I.$$

Similarly, the per-user profit of the platform is:

$$\tilde{\pi}^P(p^I, p^P) = (p^P - c^P)\tilde{x}^P(p^I, p^P).$$

And its total profit is equal to:

$$\Pi^P(\alpha, p^I, p^P) = \alpha\tilde{\pi}^P(p^I, p^P) - f^P.$$

From Assumption 1, for all prices, we have

$$\frac{\partial \Pi^I}{\partial \alpha} = (p^I - c^I)(\tilde{x}(p^I, p^P) - x(p^I)) \leq 0, \tag{1}$$
$$\frac{\partial \Pi^P}{\partial \alpha} = \tilde{\pi}^P(p^I, p^P) \geq 0. \tag{2}$$

Direct effects of affiliation are intuitive. For any given prices, a higher affiliation decreases the profit of the incumbent and increases the profit of the platform.

4 The static game: Equilibrium prices and the competition in the market effect

In this section, we derive the optimal prices in the static game, that is considering the proportion of affiliated consumers (α) as given. We show that when α increases competition in the market intensifies.

4.1 Best reply functions

Firms are competing in prices. One important feature of this price game is that the incumbents serve all the consumers, affiliated and non-affiliated, but it cannot price discriminate between the two groups.

The structure of this competitive problem has been studied by Valletti et al. (2002) and Gautier and Wauthy (2010) in the context of universal service obligations. These papers identify a market sharing pure strategy equilibrium in which the incumbent's best reply function $\phi^I(p^P, \alpha)$ is defined as follow[10]:

$$\phi^I(p^P, \alpha) = \underset{p^I}{\operatorname{argmax}}\, \Pi^I \text{ is the solution of } \alpha \frac{\partial \tilde{\pi}^I(p^I, p^P)}{\partial p^I} + (1-\alpha)\frac{\partial \pi^I(p^I)}{\partial p^I} = 0. \tag{3}$$

And the platform's best reply function solves:

$$\phi^P(p^I) = \underset{p^P}{\operatorname{argmax}}\ \Pi^P \text{ is the solution of } \frac{\partial \tilde{\pi}^P(p^I, p^P)}{\partial p^P} = 0. \tag{4}$$

The equilibrium prices are given by the solution of the system composed of Equations (3) and (4). Before describing the equilibrium, we establish the following result:

Lemma 1 *(1) The incumbent's best reply ϕ^I shifts downward when α increases.*
(2) The platform's best reply is independent of α.

The platform exerts a competitive pressures on the incumbent, and, as affiliation to the platform increases, this pressure becomes stronger. As a consequence, the incumbent is more aggressive in the price game. This downward shift in the incumbent's best reply function captures the competition in the market effect.[11] Figure 1 illustrates the lemma.

4.2 Equilibrium prices

The static price equilibrium $(p_s^I(\alpha), p_s^P(\alpha))$ is the solution of

$$p_s^I(\alpha) = \phi^I(p_s^P(\alpha), \alpha) \text{ and } p_s^P(\alpha) = \phi^P(p_s^I(\alpha)). \tag{5}$$

A direct consequence of Lemma 1 is that:

Lemma 2 *The equilibrium prices are decreasing in α.*

[10] Gautier and Wauthy (2010) define the conditions for the existence of an interior equilibrium. We suppose that these conditions are satisfied.

[11] Instead of considering a single incumbent, we can consider a competitive fringe of N incumbents, whose representative member is denoted F . For each incumbent i, we can construct, as we did in Equation 3, a price best reply $\phi_i^I(p^P, p_{-i}^P, \alpha)$ where p_{-i}^P is the price vector of the $N-1$ other incumbents than i. Assuming symmetric incumbents, we can solve the system composed of these N best reply functions and construct a fringe best reply $\phi_F^I(p^P, \alpha)$ which gives the optimal fringe price as a function of the platform price and the affiliation. This fringe best reply satisfies condition (1) of Lemma 1, therefore, we can reproduce identically our analysis for the case of $N > 1$ incumbents.

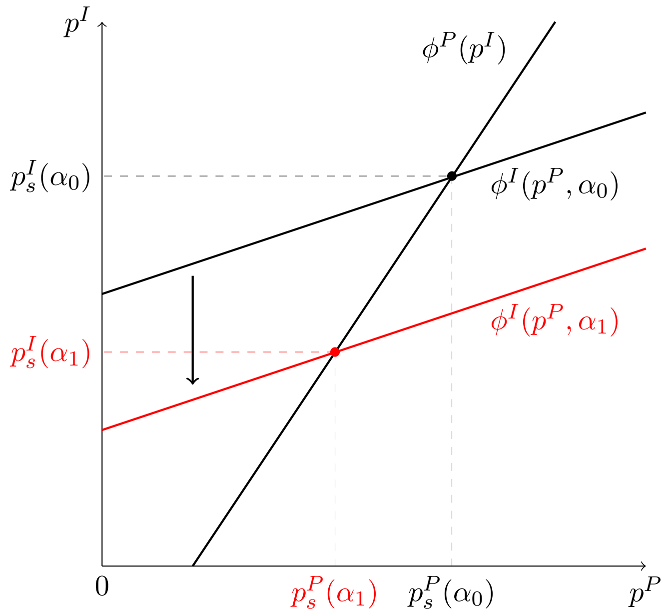

Figure 1: Competition *in* the market effect of affiliation, when $\alpha_0 < \alpha_1$

An increase in the number of affiliated consumers α intensifies competition and result in lower prices for both the incumbents and the platforms. It captures the idea that the incumbents will be more aggressive when the platform extends its market coverage and this claim is supported by empirical evidences.[12]

4.3 Comparative statics

Let us denote the equilibrium profits when the affiliation is equal to α by $\Pi^I_s(\alpha) = \Pi^I_s(\alpha, p^I_s(\alpha), p^P_s(\alpha))$ and $\Pi^P_s(\alpha) = \Pi^I(\alpha, p^I_s(\alpha), p^P_s(\alpha))$. For standard demand functions satisfying Assumption 1, the following comparative static results hold true[13]:

Lemma 3 *(1) $\Pi^I_s(\alpha)$ is decreasing in α and (2) $\Pi^P_s(\alpha)$ is concave in α.*

An increase in α intensifies competition and leads to lower prices. For the incumbent, this ultimately leads to lower profits. For the platform, there is in addition an other effect at play, as a larger affiliation basis increases the platform's market size. For this reason, we can show that the platform's profit is concave in α. For our analysis, we will assume the following:

Assumption 2 *(1) For all $\alpha \in [0,1]$, $\Pi^P_s(\alpha)$ increases in α and (2) $\Pi^P_s(1) > 0$.*

[12] See for instance Zervas *et al.* (2017).

[13] These results are standard and already proved in Valletti *et al.* (2002).

5 Affiliation to the platform

In the sequel, we consider a continuous time dynamic model, where affiliation to the platform changes over time. At each time t, firms set their prices $(p^I(t), p^P(t))$, consumers buy products, firms collect profits, and new consumers affiliate to the platform. Our dynamic model builds up on the static model developed in the previous section. The model uses reduced forms for the profit functions satisfying Lemma 3 and Assumption 2.

We model affiliation as a dynamic process. It is well documented that platforms are progressively building their user base and often, to be successful, they must reach a critical mass of users. Platforms are building on network effects, and to increase participation, platforms use advertising, they count on words of mouths or they try to convince consumers by adopting low introductory prices. Our model tries to capture these features.

We follow Evans and Schmalensee (2010) and we suppose that there is a target population of size 1 but the entire target population is almost never well-informed at the launch of the platform i.e. the actual number of users is different from the potential number of users. The evolution of the user base is a dynamic process, users progressively learn about the existence of the platform either from advertising or word of mouth, and may decide to affiliate with it.[14]

We denote by $\alpha(t)$ the share of affiliated consumers at date t and we suppose that $\dot{\alpha}(t)$ depends on $\alpha(t)$ and $p^P(t)$:

$$\frac{d\alpha}{dt} \equiv \dot{\alpha} = A(\alpha, p^P),$$

with $A(.)$ decreasing in p^P.[15]

Our analysis follows the Lanchesterian tradition for modeling marketing rivalry and adapt it to the case of affiliation. Originally used to model military combat, Kimball (1957) introduced the Lanchester model into the economic field and it was further developed by Case (1979) and Little (1979). It typically refers to a competition for market dominance where advertising plays a central role, primarily influencing the customers of competing companies.

In our context, only the platform can attract customers who are not affiliated yet, which are in proportion $(1 - \alpha)$. To do this, the platform does not use advertising but prices, and we suppose that lower prices boost affiliation.

[14]See also Casey and Töyli (2012) for a similar formulation of a dynamic model of affiliation.

[15]In their model, Evans and Schmalensee (2010) consider that the price p^P is fixed.

In our model, we specify the affiliation process as follow:

$$A(\alpha, p^P) = (1-\alpha)\beta(p^P) \geq 0, \tag{6}$$

where $\beta(p^P)$ is a positive decreasing convex function of p^P.[16] In this specification, by reducing its price, the platform affects both the demand from affiliated consumers and the consumers' willingness to affiliate.

Our affiliation process has two main properties. First, there is an autonomous affiliation process that does not directly depend on time but on the affiliation base itself. To illustrate, suppose that $\alpha(0) = \alpha_0$, that is the platform is not starting from scratch at date $t = 0$. and the platform commits to a charge constant price $\bar{p}^I$ for all periods. The platform attracts $(1-\alpha)\beta(\bar{p}^P)$ consumers in the next period $t + dt$. Solving the simple differential equation that represents the affiliation process leads to the following path for the platform's user basis $\alpha(t) = 1 - (1-\alpha_0)\, e^{-\beta(\bar{p}^P)t}$ and $\alpha(t)$ is increasing and concave and converges to 1 in infinite time. Second, the platform can speed up affiliation by lowering its price. But this process has decreasing marginal returns to scale: additional members become less and less sensitive to price reductions.

This specification is chosen to limit structural effects in membership dynamics that would be too advantageous for the platform. Moreover, the sensitivity of affiliation to price should be bounded above to keep an interior solution for the dynamic problem to come.

To provide micro-foundations for our affiliation model, we define the net gain from affiliation as the utility differential between affiliation and non-affiliation: $\Delta U = U_a(p^P, p^I) - U_n(p^I)$ and ΔU decreases in p^P. Suppose that there is a non-pecuniary costs κ_j to individual j of participating to the platform. This cost represents the individual preferences for the platform's features such as design, ease of use, privacy, security or advertising. Given that, an individual j affiliates if:

$$\Delta U - \kappa_j \geq 0.$$

Suppose that κ is randomly distributed across individuals and at every period on a continuous support $[\underline{\kappa}, \overline{\kappa}]$ according to $g(\kappa)$. Time variations capture the fact that consumer preferences may evolve with exposure or experience. We can define κ^* such that $\Delta U = \kappa^*$. Then, the probability of affiliation at each time t is given by is $G(\kappa^*) = \int_{\underline{\kappa}}^{\kappa^*} g(\kappa) d\kappa$. Hence, at time t,

[16] In the quadratic utility function example in Appendix B, we suppose that $\beta(p^P) = \zeta - \beta p^p$, with $\beta > 0$.

given that there are still $(1-\alpha)$ unaffiliated users, affiliation is equal to:

$$\dot{\alpha} = (1-\alpha)\, G(\kappa^{*}) = (1-\alpha)\, G(\Delta U).$$

Hence we can define $\beta = G(\Delta U)$. In the Singh and Vives example ΔU is decreasing and convex in p^P. Any distribution function that preserves this convexity can be used to define β.

Finally note that our model excludes the possibility that the incumbent uses limit pricing to slowdown consumers' affiliation to the platform. With a fringe of N incumbents (see footnote 11), none of them has an impact on the incumbents' price. Hence, if we consider that affiliation depends not on platform's price but on its price relative to the incumbent's price, for instance if we specify $\beta(\Delta p)$, with $\Delta p = p^P - p^I_F$, the incumbents cannot influence affiliation by charging a low price as $\frac{\partial p^I_F}{\partial p^I_i} = 0$. Hence, without any strategic effect on the incumbent's side, this formulation is qualitatively equivalent to the formulation above.

6 Dynamic pricing

We consider the dynamic pricing game where, at each time t, firms compete in prices. As it does not have an influence on the affiliation process, the behavior of the incumbent is unchanged compared to the static version of the model and captured by the best reply function ϕ^I, which is a function of the current affiliation $\alpha(t)$.

On the other hand, the platform knows that, by lowering its current price, it speeds up affiliation to the platform and increases its future profits. So there will be an additional effect on the price best-reply of the platform and this effect will capture the competition for the market. Formally, the platform will choose its price $p^P(t)$ by solving the following dynamic program $(\mathcal{P})$:

$$\max_{p^P(t)} \int_0^{\infty} e^{-rt}\Pi^P(\alpha, p^I, p^P)dt \quad \text{subject to } \dot{\alpha} = (1-\alpha)\beta(p^P). \tag{7}$$

We assume that $\alpha(0) = \alpha_0$ with $\alpha_0 \to 0$, that is the platform is not starting from scratch at date $t = 0$ but it has a small user base at date 0. The problem is an optimal control problem where the price $p^P(t)$ is the control variable and $\alpha(t)$ the state variable. The prices at period t are determined by solving the system of best reply functions. For the incumbent, the best reply is given by $\phi^I(p^P, \alpha)$, for the platform by the above maximization program.

6.1 Benchmark: The full myopic case

To create a benchmark, we will consider first the case of a myopic platform. A myopic platform does not take into account the impact on its price on the affiliation process. We solve this case by solving $(\mathcal{P})$ and setting the discount rate r arbitrarily to zero. In such case, we ignore the dynamics of affiliation as a state constraint in the above maximization problem. In this case, the platform's price best reply is equivalent to the static one ϕ^P and, for any given α, the equilibrium prices are identical to the static model and given in Equation (5).

The resulting price equilibrium path is now $\{(p_m^I(t), p_m^P(t))\}_{t=0}^{\infty}$ such that, for any level of $\alpha = \alpha(t)$ driven by the affiliation process, we have

$$p_m^I(t) = p_s^I(\alpha(t)) \quad \text{and} \quad p_m^P(t) = p_s^P(\alpha(t)).$$

Denote $\alpha_m(t)$ the induced affiliation equilibrium path from $\dot{\alpha}(t) = (1-\alpha(t))\beta(p_m^P(t))$ with $\alpha(0) = \alpha_0$.

Proposition 1 *In the full myopic case, prices are a decreasing function of time under the direct effect of affiliation dynamics i.e.*

$$\dot{p}_m^I(t) = \frac{\partial p_s^I(\alpha_m(t))}{\partial \alpha}\dot{\alpha}_m(t) < 0 \text{ and } \dot{p}_m^P(t) = \frac{\partial p_s^P(\alpha_m(t))}{\partial \alpha}\dot{\alpha}_m(t) < 0.$$

Incumbents' profits are decreasing in time. The market is asymptotically fully covered by the platform.

The first two results simply follow from the joint application of those of the lemma 3 and the positive sign of the dynamics in Equation (6).

In the myopic case, we have the following properties:

Corollary 1

1. *The platform's price is always above its marginal cost $p_s^P(\alpha) \geq c^P$.*
2. *There exist $\bar{\alpha}$ such that $\bar{\alpha}(p_s^P(\bar{\alpha}) - c^P)\tilde{x}^P() = f^P$.*
3. *It exists a date $\bar{t}$ such that $\alpha(\bar{t}) = \bar{\alpha}$. The platform has a negative profit for $t < \bar{t}$ and positve after.*

For any α, the firms play the static game price equilibrium. However, for the platform, a low affiliation may not be sufficient to recover its fixed cost. But, as the profit Π^P is increasing in α, and it is positive for $\alpha = 1$ (Assumption 2), there exists a threshold value $\bar{\alpha}$ such that the platform has a negative profit for α below and a positive one for α above. This implies that the platform finds it profitable to enter the market if the initial losses realized during $[0, \bar{t}]$ are more than compensated by the profits realized afterward, that is if:

$$\int_0^{\bar{t}} e^{-rt} \Pi^P(\alpha, p_m^I(t), p_m^P(t))dt \leq \int_{\bar{t}}^{\infty} e^{-rt} \Pi^P(\alpha, p_m^I(t), p_m^P(t))dt. \quad (8)$$

We illustrate the benchmark case with the following figures. Figure 2 illustrates the myopic affiliation path, Figures 3 and 4, the price paths and the profit paths. To construct the figures, we consider a symmetric case where where $c^I = c^P$ and $f^I = f^P$.

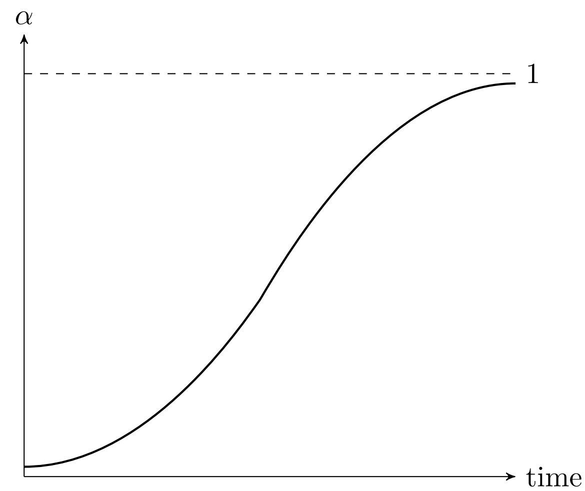

Figure 2: The myopic case: affiliation path

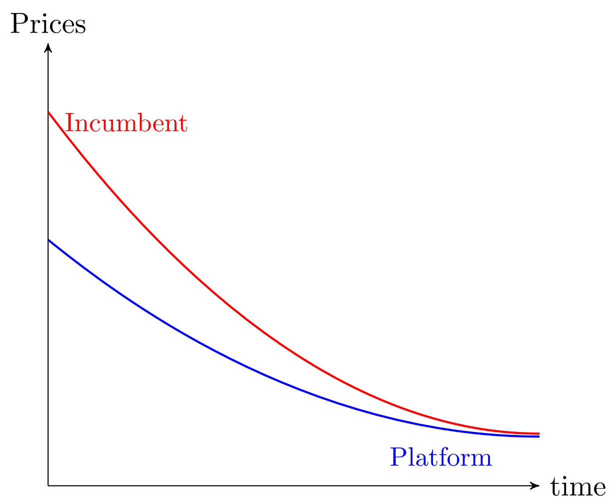

Figure 3: The myopic case: price paths

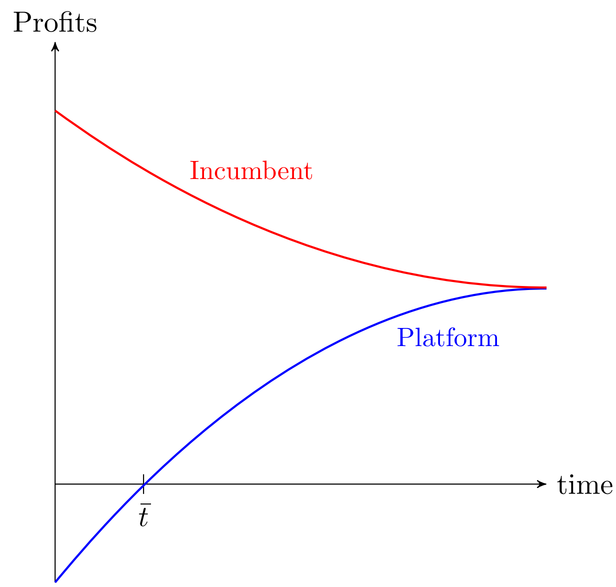

Figure 4: The myopic case: profit paths

6.2 Dynamic pricing: the forward looking case

We now solve the program $(\mathcal{P})$ with $r > 0$ and denote $\{(p_*^I(t), p_*^P(t))\}_{t=0}^{\infty}$ the solution. In that case, the platform takes into account the impact of its pricing on affiliation, i.e. it competes both *in* and *for* the market. This dynamic pricing problem is an autonomous optimal control problem. We associate to the affiliation process (Equation 6) a co-state variable λ, which is the shadow value of affiliation. We look after the steady state equilibrium $(\alpha, \lambda) = (1, \lambda^\infty)$ and the price trajectory leading to this equilibrium. This consists in specifying the pace of the state α and the co-state variables λ as prices directly depend on these two variables.

Proposition 2 *There exists a steady state equilibrium ($\alpha = 1$ and $\lambda = \lambda^\infty$) with $\lambda^\infty > 0$ and $\dot{\lambda} < 0$. The equilibrium exhibits the saddle point property.*

We now analyze the properties of the equilibrium path, and in particular the implication of a positive and decreasing shadow value of affiliation λ. First we characterize the platform's best reply in the dynamic case, denoted $\psi^P(p^I, \alpha, \lambda)$.

Lemma 4 *The platform dynamic best reply ($\psi^P(p^I, \alpha, \lambda)$) is defined by*

$$p^P = \psi^P(p^I, \alpha, \lambda) : \frac{\partial \tilde{\pi}^P\left(p^I, p^P\right)}{\partial p^P} = -\lambda \frac{1-\alpha}{\alpha} \beta'\left(p^P\right) > 0,$$

and $\lambda \geq \lambda^\infty = \frac{\tilde{\pi}^P(p_s^I(1), p_s^P(1))}{r + \beta(p_s^P(1))} > 0.$

Compared to the static case, the platform's best reply has an additional term, capturing the competition for the market effect. Given that $\lambda > 0$, this additional term is negative which means that, for any given α, the platform is more aggressive in the price game than in the static game. As a consequence, the platform's best reply $\psi^P(p^I, \alpha, \lambda)$ defined in the previous lemma is lower than the static best reply:

$$\psi^P(p^I, \alpha, \lambda) < \phi^P(p^I).$$

Affiliation, as a dynamic process, strengthens price competition at all periods. This result is driven by the shadow value λ the platform places on the affiliation dynamics. Because affiliation is dynamic, a higher affiliation increases the future profits of the platform for a given price. As a result, the platform can sacrifice its current profits by reducing its price to attract users, in exchange for higher future profits, and this value is captured by λ. This shadow value of affiliation is decreasing in time and converges to the limit value λ^∞ which is a perpetuity based on the flow of platform's gross profit per consumer in the long run when all consumers are affiliated. This profit is discounted at the discount rate r plus the attraction rate $\beta\left(p_s^P(1)\right)$ evaluated at the long run price when the platform has affiliated all consumers.[17]

At each period t, optimal dynamic prices are the fixed point of the best replies, given by

$$p^P = \psi^P\left(p^I, \lambda, \alpha\right) \text{ and } p^I = \phi^I\left(p^P, \alpha\right),$$

where $\phi^I(p^P, \alpha)$ is the same best-reply as in the static game, i.e Equation (3) and $\psi^P(p^I, \alpha, \lambda)$ is defined in Lemma 4. Consequently, for any α, dynamic prices are lower than in the static game.

In the dynamic game α and λ depend on time. From that we can compute time-dependent prices: $p_*^I(t)$ and $p_*^P(t)$. We can show these prices are lower compared to the myopic benchmark.

Proposition 3 *For all firms, the dynamic equilibrium price paths are always lower than in the myopic setting i.e.*

$$p_*^P(t) \leq p_m^P(t) \quad \text{and } p_*^I(t) \leq p_m^I(t) \text{ at each date } t.$$

[17] $\beta\left(p_s^P(1)\right)$ is equal to $\frac{\dot{\alpha}}{1-\alpha}$ representing the level of new consumers attracted among the share of non affiliated consumers. It is similar to a hazard rate of the affiliation process.

And a straightforward consequence of lower prices is that affiliation to the platform is faster compared to the myopic case.

Proposition 4 *The equilibrium affiliation path is always higher than the myopic one:*

$$\alpha_* (t) \geq \alpha_m (t) .$$

We illustrate Proposition 4 on the following figure.

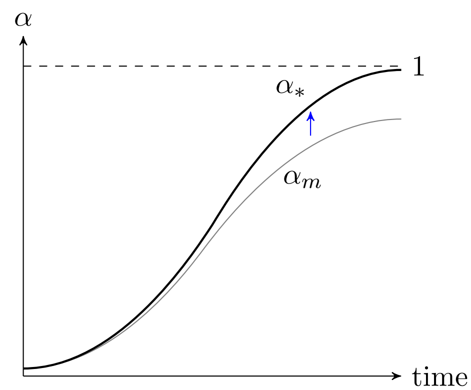

Figure 5: Affiliation paths: dynamic *vs* myopic

Regarding the platform's profit, there are two effects at play. First, for any given α, the platform has a lower profit compared to the static case, as prices depart from the Nash prices in the static game. But, second, affiliation is faster and affiliation increases the profits. Because of these two opposite effects, it is not possible to establish if the platform reaches the break-even point before or after $\bar{t}$.

6.3 The competition for the market effect

In our model, we can identify a competition in the market effect and a competition for the market effect. The competition in the market effect is captured by the downward shift in the incumbent's best reply as affiliation (α) increases. The competition for the market effect is captured by the additional term, $-\lambda \frac{1-\alpha}{\alpha} \beta' \left(p^P\right)$, in the incumbent's best reply function (Lemma 4). We now discuss the relative importance of these two effects over time and the consequences on the price paths.

In Lemma 1, we have shown that competition in the market intensifies, the incumbent becomes more aggressive in the price game, as affiliation grows. Similarly, we can show that the competition for the market effect decreases over time. More precisely, the platform's best reply function shifts upward as time goes by.

Lemma 5 *The platform's best reply* $\psi^P(p^I, \alpha, \lambda)$ *increases over time and converges to* $\phi^P\left(p^I\right)$ *when* $\alpha \to 1$.

The lemma shows that the competition for the market effect is decreasing over time. This effect is, quite logically, rather strong at the early stages of the platform's development. And, as affiliation increases, this effect fades out and the platform becomes relatively less agressive in the price game. We illustrate that in the following figure.

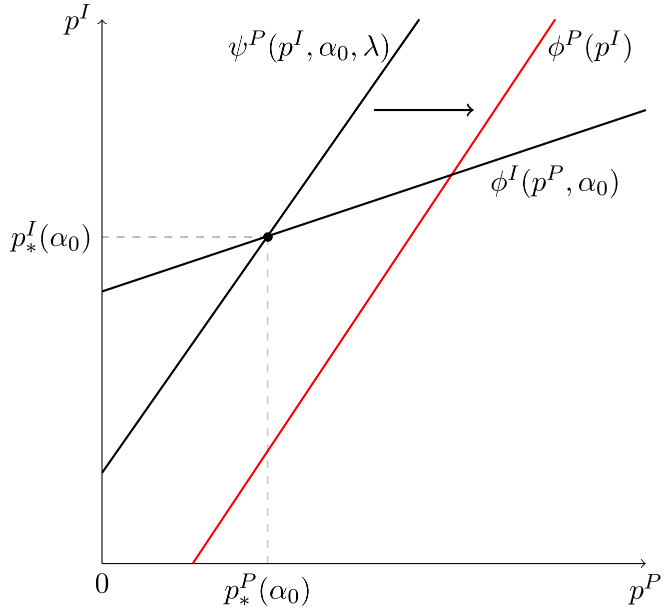

Figure 6: Competition *for* the market effect of affiliation, when $\alpha_0 < 1$

But, as affiliation grows over time, competition for the market is progressively replaced by the competition in the market. The the two effects go in opposite direction as α increases. The competition for the market effect decreases, making the platform less aggressive in the price game. The competition in the market effect increases, making the incumbent more aggressive in the price game.

Over time, diminishing competition for the market drives price up while intensified competition in the market drives price down. These two conflicting forces may lead to non-monotonic price paths. We summarize that in the following corollary and figure.

Corollary 2 *The platform's price path is not necessarily decreasing in time.*

To illustrate, we use the example developed in Appendix B. In this example, we suppose that β' is constant, and one can show that:

$$\lim_{\alpha \to 0} \dot{p}_*^P(t) = +\infty \text{ and } \lim_{\alpha \to 1} \dot{p}_*^P(t) < 0.$$

So it exits at least one date for which the platform's price path changes its slope. The price is first increasing as competition for the market is less intense, then it decreases as competition in the market intensifies. The following figure illustrates both lower prices compared to the myopic game (Proposition 3) and non-monotonic price paths.

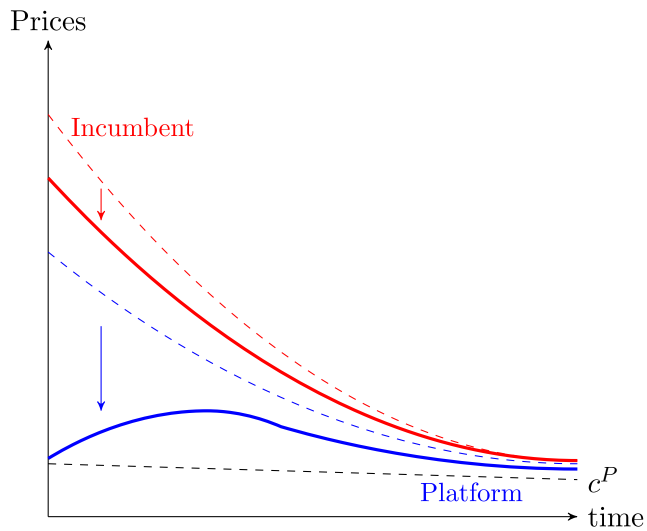

Figure 7: Affiliation paths: dynamic *vs* myopic

7 Extensions

We extend our analysis in two directions. First, we introduce the possibility that the market tips and the incumbent exits the market. Second, we consider that the platform is two-sided and should affiliate both consumers and service providers.

7.1 Market tipping

Assume that $\Pi^I(1, p_s^I(1), p_s^P(1)) < 0$. This implies that, in the long run, when the platform covers all the market, the incumbent is making losses. In this configuration, the incumbent will exit the market at some time T and the platform will monopolize it, i.e. the market will tip. In this extension, we analyze this market tipping configuration and we show how it impacts pricing by the platform.

For that, we need to identify a new individual demand function that apply when the incumbent exits the market. Let us define this monopoly demand function by $x_j(p^P)$ and suppose that it satisfies $x_j^P(p^P) \geq \tilde{x}_j^P(p^P, p^I)$ and the conditions of Assumption 1. When the incumbent exits the market, the profit of the platform is defined as $\tilde{\Pi}^P(\alpha, p) = \alpha\pi^P(p) - f^P$ with $\pi^P(p) = (p - c^P)x^P(p)$. We denote p^m the static monopolistic solution such that $p^m = \arg\max_p \pi^P(p)$. If the incumbent exits the market at date T, the platform solves the following program:

$$\max_{p^P} \int_0^T e^{-rt}\Pi^P(\alpha, p^I, p^P)dt + \int_T^\infty e^{-rt}\tilde{\Pi}^P(\alpha, p^P) \quad \text{subject to } \dot{\alpha} = (1-\alpha)\beta(p^P).$$

In this problem we need to identify the incumbent's decision to exit the market and exit should be irreversible.

We suppose that if the platform manages to affiliate a fraction $\tilde{\alpha}$ of the consumers such that $\Pi^I(\tilde{\alpha}, p_s^I(\tilde{\alpha}), p_s^P(\tilde{\alpha})) = 0$, then the incumbent exits the market. This threshold level corresponds to a market configuration where the incumbent has no possibility to be profitable in the long run.

The exit time T is defined by $\alpha(T) = \tilde{\alpha}$. Given that, the platform's problem can be decomposed in two sub-problems. First, we can solve the pricing problem when the incumbent has exited the market, that is

$$\max_p \int_T^\infty e^{-rt}\tilde{\Pi}^P(\alpha, p)dt \quad \text{subject to } \dot{\alpha} = (1-\alpha)\beta(p) \text{ and } \alpha(T) = \tilde{\alpha}.$$

Let us define the platform monopoly price by $p^M(t)$, the solution will be such that: for $\alpha \geq \tilde{\alpha}$,

$$p^M(t) : \frac{\partial \pi^P(p)}{\partial p} + \lambda \frac{1-\alpha}{\alpha}\beta'(p) = 0$$

That is the platform will charge a price below the monopoly price until it reaches the full market coverage and this discount decreases over time. As the horizon is infinite, we can define by $\mathfrak{P}(\tilde{\alpha})$ the optimal value of this profit flow starting at $t = 0$ such that

$$\mathfrak{P}(\tilde{\alpha}) = \int_0^\infty e^{-rt}\tilde{\Pi}^P(\alpha^M(t), p^M(t))dt,$$

where $\alpha^M(t)$ is the user share path when $\alpha^M(0) = \tilde{\alpha}$ and based upon the a monopolistic price path. The monopolistic solution implies again the existence of a co-state λ^M such that the pair (α^M, λ^M) exhibits the saddle-point property but now with $\lambda_M^\infty = \frac{\pi^P(p^m)}{r+\beta(p^m)} > \lambda^\infty$ as defined in the Lemma (4), and with $\dot{\lambda}^M < \dot{\lambda} < 0$. Consequently, compared to the duopolistic dynamic game, the initial optimal value $\lambda^M(0) > \lambda(0)$. Note that for any $\alpha^M(0) = \tilde{\alpha}$, by definition we have $\mathfrak{P}'(\tilde{\alpha}) = \lambda^M(0)$.

In a second step, we can solve the following problem:

$$\max_{p^P(t),\ T} \int_0^T e^{-rt}\Pi^P(\alpha, p^I, p^P)dt + e^{-rT}\mathfrak{P}(\tilde{\alpha}) \quad \text{subject to } \dot{\alpha} = (1-\alpha)\beta(p^P) \text{ and } \alpha(T) = \tilde{\alpha}.$$

Let us denote $\hat{\lambda}$ the co-state variable for this sub-problem. The difference with the problem $(\mathcal{P})$ is that the horizon is not infinite and it is a decision variable as well as the fraction $\tilde{\alpha}$ of the platform's users at the date. However, the price solution is still defined by applying the best replies defined in Equation 3 and in Lemma (4), complemented by a new transversality condition[18] for $\tilde{\alpha}$:

$$\tilde{\alpha}^* : \hat{\lambda}(T) = \mathfrak{P}'(\tilde{\alpha}^*) = \lambda^M(0).$$

The terminal value of $\hat{\lambda}(T)$ is now based on the overall monopoly profit obtained by adding attracting a user in date T^* i.e. $\mathfrak{P}'(\tilde{\alpha}^*)$ which is higher than without market tipping at the same date T^* as

$$\hat{\lambda}(T^*) = \lambda^M(0) > \lambda(0) > \lambda(T^*).$$

Before market tips, the platform's price best reply is not changed compared to the previous case. The only change is the value of the co-state variable λ and the previous analysis have shown that λ is higher, as the platform can expect higher profits from time T. Therefore, we have a stronger competition for the market effect, that will accelerate market tipping and

[18] There is also the following condition for the optimal transition date T^* :

$$\Pi^P(\tilde{\alpha}^*, p_T^I(T^*), p_T^P(T^*)) + \mathfrak{P}'(\tilde{\alpha}^*)(1-\tilde{\alpha}^*)\beta(p_T^P(T^*)) - r\mathfrak{P}(\tilde{\alpha}^*) = 0.$$

lead to lower prices before T.

When the incumbent exits, we still have a higher λ but there is no more competition in the market. This competition for attracting consumer is less intense over time ($\dot{\lambda} < 0$) and the platform price converges to the monopoly price p_M^I.

We summarize this in the following:

Corollary 3 *With market tipping, competition for the market is fiercer before market tips.*

7.2 Two-sided platform

In our main analysis, the platform provides services with the same technology than the incumbent, eventually allowing for different marginal and fixed costs. In many sectors, platforms act as matchmakers and rely on external suppliers for providing services, hosts in the tourism industry, drivers for ride-sharing. To satisfy the demand, the platform needs to attract these external suppliers. In other words, a platform needs to affiliate users on both the demand and the supply side.

Suppose that, there is a continuum $[0, 1]$ of potential service providers, a fraction a of them being affiliated to the platform at a given time. An affiliated supplier provides a volume of service $y\left(r^P\right)$ when the platform offers a price r^P, with $y' > 0$ and $y'' > 0$.[19] To provide services to clients, the platform must ensure that, at each date, supply is equal to demand:

$$ay\left(r^P\right) = \alpha\tilde{x}^P(p^I, p^P). \tag{9}$$

We suppose that affiliation on the supply side follows a dynamic process, similar to the demand side. In particular, a higher fee r^P for service increases the proportion of affiliated suppliers:

$$\dot{a} = (1 - a)\gamma(r^P), \tag{10}$$

with $\gamma(.)$ being a positive increasing convex function.

To avoid unnecessary complications, we suppose that the platform has, as before, a marginal cost c^P, eventually normalized to zero, a fixed cost $f^P > 0$ and pays $ar^P y\left(r^P\right)$ to the suppliers for their service provision. The profit of the platform can be written as:

$$\Pi^P(\alpha, p^I, p^P, a, r^P) = \alpha\tilde{\pi}^P(p^I, p^P) - ar^P y\left(r^P\right) - f^P.$$

[19] In this model, there is no competition between the platform and the incumbent on the supply side.

The platform's dynamic problem is given by ($\mathcal{P}$) to which we add the supply-side affiliation process (Equation 10) and the spot market equilibrium condition (Equation 9).

As $y\left(r^P\right)$ is inversible and denoting its inverse function by $R(\cdot)$, which is also increasing and convex in its argument, and combining with the market equilibrium condition (Equation 9), we can write:

$$r^P = R\left(\frac{\alpha}{a}\tilde{x}^P(p^I, p^P)\right). \tag{11}$$

To simplify notations, let us call $Y = \frac{\alpha}{a}\tilde{x}^P(p^I, p^P)$, and we have $r^P = R(Y)$.

In order to capture the price effects of adding a second side to the platform, we look how the price best response given in Lemma 4 is modified. For the analysis, we add a new co-state variable μ for the affiliation on the supply-side.

Lemma 6 *The platform dynamic best reply ($\hat{\psi}^P(p^I, \cdot)$) is defined by*

$$p^P = \hat{\psi}^P(p^I, \cdot) : \frac{\partial \tilde{\pi}^P\left(p^I, p^P\right)}{\partial p^P} = -\lambda \frac{1-\alpha}{\alpha}\beta'\left(p^P\right) - \left[\mu \frac{1-a}{a}\gamma'\left(R\left(Y\right)\right) - \frac{\alpha}{a}\tilde{x}^P\right]\frac{\partial \tilde{x}^P}{\partial p^P}R'\left(Y\right),$$

and $\mu \leq \mu^\infty = \frac{\tilde{x}^P(\hat{p}^I_\infty, \hat{p}^P_\infty)^2}{r+\gamma(\tilde{x}^P(\hat{p}^I_\infty, \hat{p}^P_\infty))}R'\left(\tilde{x}^P(\hat{p}^I_\infty, \hat{p}^P_\infty)\right)$ *with* $\mu^\infty > 0$.

Compared to the previous case, there are two additional terms in the best reply. When the platform decreases its price, it increases the demand from affiliated users. To satisfy this demand, the platform must increase the price r^P paid to the supply side, with two consequences. First, a higher price boosts affiliation of suppliers and this effect reinforces the competition for the market effect. Second, it increases the cost for the platform, partially offsetting the positive impact of a higher demand on the consumer side.

The impact on affiliation is captured by the term $-\mu\frac{1-a}{a}\gamma'\left(R\left(Y\right)\right)\frac{\partial \tilde{x}^P}{\partial p^P}R'\left(Y\right)$, which is positive when $\mu \geq 0$. Therefore, if $\lambda, \mu \geq 0$, the impact of affiliation on both sides tend to strengthen price competition at all periods and makes the platform even more agressive. The supply side affiliation strengthens the competition for the market effect.

But, at the same time, the platform increases the price paid to the suppliers. This positive impact on costs is captured by the last term ($\frac{\alpha}{a}\tilde{x}^P\frac{\partial \tilde{x}^P}{\partial p^P}R'\left(Y\right) < 0$). An increasing demand implies a higher price on the supply side and this negative supply cost effect countervails the impacts of affiliation. And, overall, the global impact on the platform's behavior depends on the balance between these two opposite effects.

At the steady state, the competition for the market effect vanishes and only remain the interplay between prices on the two sides i.e. a lower price on the consumer side increase the cost on the supplier side. As a consequence, we have higher prices in the steady state compared to the previous case.

Corollary 4 *With two-sided markets, at the steady state we have*

$$\hat{p}_\infty^P > p_\infty^P \text{ and } \hat{p}_\infty^I > p_\infty^I.$$

While this effect does not come as a surprise as, in our framework, the addition of a second side increases the cost of the platform. More importantly, we show that there is an interplay between the prices on the two sides and that this affects the pricing strategy of the platform.

8 Conclusion

In this paper we have modelled the competitive dynamics between a digital platform and a vertically integrated incumbent, capturing two distinct but interacting forces: competition for the market and competition in the market. Using the example of ride-sharing platforms such as Uber and Lyft, we analyze how platforms enter traditional sectors with innovative, asset-light business models and gradually reshape market structures. Despite offering comparable or superior services, platforms face initial barriers to consumer adoption due to the requirement of user affiliation-downloading the app, creating an account, and building trust i.e. to compete, a platform needs to build up its user base.

To overcome these hurdles and build a critical mass of users, platforms often operate at a loss and relying on investor funding. This strategy, rooted in the competition for the market, is aimed at rapidly growing the affiliated user base. As affiliation expands, platforms become more competitive and pose a greater threat to incumbents, who must respond by lowering prices, leading to intensified competition in the market.

Our dynamic model formalizes this process and identify the two competitive effects. Competition for the market, to affiliate users to the platform, changes the platform's best reply in the price game and we show that, as affiliation increases, the platform becomes relatively less agressive (its price best-reply shifts upward). To the contrary, when affiliation increases, competition for the market intensifies. We identify this effect by showing that the incumbent is becoming relatively more aggressive in the price game (its price best-reply shifts downward).

Our dynamic model highlights a key finding: the platform's pricing path is non-monotonic. Initially, prices are low to stimulate user affiliation. Over time, as the user base grows and the need to incentivize new affiliation diminishes, the platform may increase its price. However, with a larger user base comes stronger market competition, prompting another round of price reductions.

Ultimately, our analysis reveals that the platform's profitability trajectory is shaped by the interplay of these two opposing forces. Competition for the market dominates at early stages, while competition in the market becomes the driving force as the platform matures. This framework helps explain the pricing strategies, growth patterns, and long-run profit dynamics of digital platforms across various sectors, from ride-sharing to accommodation, delivery, and beyond.

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A Proofs

Proof of Lemma 1. This result has been established by Valletti et al. (2002) and Gautier and Wauthy (2010). The idea is that an incumbent's best reply ϕ^I is computed along the two branches of the profit function, corresponding to two different market structures. So to simplify let $\phi^I(\alpha) = \phi^I(p^I, p^P, \alpha)$.

Define $\tilde{\phi}^I = \text{argmax}_{p^I} \tilde{\pi}^I(p^I, p^P)$, the incumbent's best reply corresponding to market configuration where there is competition the incumbent and the platform (duopoly case when $\alpha = 1$). Similarly define $\hat{\phi}^I = \text{argmax}_{p^I} \pi^I(p^I)$, the price decision corresponding to a market where the platform is absent and there is only an incumbent (when $\alpha = 0$). Assumption 1 implies that:

$$\phi^I(1) = \tilde{\phi}^I \leq \phi^I(\alpha) \leq \hat{\phi}^I = \phi^I(0). \tag{12}$$

Furthermore, if we take the total derivative of $\phi^I(\alpha)$ with respect to α, we have:

$$\frac{\partial \phi^I(\alpha)}{\partial \alpha} = -\frac{\partial^2 \pi^I / \partial p^I \partial \alpha}{\partial^2 \pi^I / \partial \left(p^I\right)^2} = -\frac{\left(p^I - c^I\right)\left(\frac{\partial \tilde{x}^I}{\partial p^I} - \frac{\partial x^I}{\partial p^I}\right) + \left(\tilde{x}^I - x^I\right)}{\partial^2 \pi^I / \partial \left(p^I\right)^2} < 0,$$

which is negative by Assumption 1. Part (2) of the lemma is obvious.

Proof of Lemma 2. Indeed since all those profits functions are concave in own prices, we have the following cost pass-through effects, for any (p^I, p^P):

$$\begin{aligned}
\frac{\partial \tilde{\phi}^I}{\partial c^I} &= -\frac{\partial^2 \tilde{\pi}^I / \partial p^I \partial c}{\partial^2 \tilde{\pi}^I / \partial \left(p^I\right)^2} = \frac{\partial \tilde{x}^I / \partial p^I}{\partial^2 \tilde{\pi}^I / \partial \left(p^I\right)^2} > 0, \\
\frac{\partial \hat{\phi}^I}{\partial c^I} &= \frac{\partial x^I / \partial p^I}{\partial^2 \pi^I / \partial \left(p^I\right)^2} > 0, \\
\frac{\partial \tilde{\phi}^P}{\partial c^P} &= \frac{\partial \tilde{x}^P / \partial p^P}{\partial^2 \tilde{\pi}^P / \partial \left(p^P\right)^2} > 0.
\end{aligned}$$

For any $\alpha \in [0, 1]$, whenever $c^P \leq c^I$, from (12) and the cost pass-through effects above, best replies are such that

$$\tilde{\phi}^P \leq \tilde{\phi}^I \leq \phi^I(\alpha) \leq \hat{\phi}^I.$$

So equilibrium prices exhibit the property that: $p_s^P(\alpha) \leq p_s^I(\alpha)$. As explaned in Vives (2001), these results are also supported by the feature that the price game is smooth supermodular in prices. Indeed, from Assumption 1:

$$\frac{\partial \Pi^P}{\partial p^P \partial p^I} = \alpha \frac{\partial^2 \tilde{\pi}^P(p^I, p^P)}{\partial p^P \partial p^I} \geq 0, \text{ and } \quad \frac{\partial^2 \Pi^I}{\partial p^I \partial p^P} = \alpha \frac{\partial^2 \tilde{\pi}^I(p^I, p^P)}{\partial p^I \partial p^P} \geq 0.$$

Moreover, from Assumption 1, firms' profits exhibits increasing differences in $A = -\alpha$.

$$\begin{aligned}\frac{\partial^2 \Pi^I}{\partial p^I \partial A} &= -\frac{\partial \tilde{\pi}^I(p^I, p^P)}{\partial p^I} + \frac{d\pi^I(p^I)}{dp^I} = -\left(\tilde{x}^I - x^I + (p^I - c^I)(\frac{\partial \tilde{x}^I}{\partial p^I} - \frac{\partial x^I}{\partial p^I})\right) \geq 0, \\ \frac{\partial^2 \Pi^P}{\partial p^P \partial A} &= 0.\end{aligned}$$

Equipped with both properties, the Topkis theorem applies (for instance see Amir, 2005) and the equilibrium is increasing in A and then non-increasing in α.

Proof of Lemma 3 (1) Using the enveloppe theorem, the total derivative of the equilibrium $\Pi_s^I(\alpha)$ is equal to $\frac{\partial \Pi_s^i}{\partial \alpha} = \tilde{\pi}^I - \pi^I$ which is negative by Lemma 2 and Assumption 1. (2) $\frac{\partial^2 \Pi_s^P}{\partial \alpha^2} = \frac{\partial \tilde{\pi}^P(p^I,p^P)}{\partial p^I} \frac{\partial \phi^I(\alpha)}{\partial \alpha} \leq 0$, by Lemma 1 and Assumption 1.

Proof of Proposition 1. Based on equilibrium prices for each level of α, the affiliation process leads to $\dot{\alpha} = (1-\alpha)\beta(p_s^P(\alpha))$. Let us define $h(\alpha) = (1-\alpha)\beta(p_s^P(\alpha))$. It is a continous function for all $\alpha \in [0,1]$ and its derivative is

$$h'(\alpha) = -\beta(p_s^P(\alpha)) + (1-\alpha)\beta'(p_s^P(\alpha))\frac{dp_s^P(\alpha)}{d\alpha},$$

which has not a constant sign. As a result, when $\alpha \to 1$, $h(1) = 0$ so $\alpha = 1$ is a stationnary equilibrium of the affiliation dynamics. Moreover as $h'(1) = -\beta(p_s^P(1)) < 0$ it is asymptotically stable. As a result, the affiliation path is locally concave when $t \to \infty$; but not necessary for all t. Applying Lemma 2 leads to $\frac{\partial p_s^I}{\partial \alpha} \leq 0$ and $\frac{\partial p_s^P}{\partial \alpha} \leq 0$. Combined to $\dot{\alpha} \geq 0$, we have the first result. Second using Equation (1), we have

$$\dot{\Pi}^I = \dot{\alpha}\left(\frac{\partial \Pi^I}{\partial \alpha} + \alpha \frac{\partial \tilde{\pi}^I(p^I, p^P)}{\partial p^P}\frac{\partial p_s^P}{\partial \alpha}\right) \leq 0.$$

Proof of Corollary 1. Point 1 is a direct consequence of the platform's best reply function given by (4). Indeed it is defined as:

$$\frac{\partial \tilde{\pi}^P(p^I, p^P)}{\partial p^P} = 0 \Leftrightarrow p^P - c^P = -\frac{\tilde{x}^P(p^I, p^P)}{\frac{\partial \tilde{x}^P(p^I,p^P)}{\partial p^P}} > 0.$$

Consequently for $\alpha \geq 0$, $\tilde{\pi}^P(p_s^I(\alpha), p_*^P(\alpha)) > 0$. As α_0 is near to 0, this implies that

$$\Pi^P(0, p_s^I(0), p_s^P(0))) = 0\tilde{\pi}^P(p_s^I(0), p_s^P(0)) - f^P = -f^P < 0.$$

Moreover when the market becomes fully duopolistic, we need to have: $\Pi^P(1, p_s^I(1), p_s^P(1))) >$

0. Indeed as $c^P \leq c^I$, if it is not true, then we have $0 \geq \Pi^P(1, p_s^I(1), p_s^P(1))) \geq \Pi^I(1, p_s^I(1), p_s^P(1)))$, and the entire industry is never profitable.

Then by composition of continuous functions of α, $\Pi^P(\alpha, p_s^I(\alpha), p_s^P(\alpha)))$ is continuous, it exists a share $\bar{\alpha}$ such that

$$\Pi^P(\bar{\alpha}, p_s^I(\bar{\alpha}), p_s^P(\bar{\alpha}))) = 0.$$

This proves the second point. Point 3 is just a direct consequence of the strictly increasing path $\alpha^*(t)$ derived from the affiliation process $\dot{\alpha}(t) = (1 - \alpha(t))\beta(p_s^P(\alpha(t)))$ which start from $\alpha(0) = \alpha_0 \to 0$ and converge to $\lim_{t \to \infty} \alpha^*(t) = 1$. As a result, we just need to define

$$\bar{t} : \alpha^*(t) = \bar{\alpha}.$$

Proof of Proposition 2.

[**1**] The platform dynamic problem exhibits a special structure: it is autonomous and discounted. Then to determine an (open-loop) Nash equilibrium $(p^I(t), p^P(t))$, one can define the current-value Hamiltonian for the platform with α as the state variable and prices as controls. Let us associate a (piecewise continuous) current-value co-state variable to $\alpha : \lambda$ for the platform. The current-value Hamiltonian is

$$H^P\left(\alpha, p^P, \lambda\right) = \alpha \tilde{\pi}^P\left(p^I, p^P\right) - f^P + \lambda(1 - \alpha)\beta\left(p^P\right).$$

We then determine the first-order conditions (that are sufficient by concavity of H^P). The Pontryagin's maximum principle implies for the platform that $\forall t \geq 0$,

$$\begin{aligned} \frac{\partial H^P}{\partial p^P} &= 0 \Leftrightarrow \alpha \frac{\partial \tilde{\pi}^P\left(p^I, p^P\right)}{\partial p^P} + \lambda(1 - \alpha)\beta'\left(p^P\right) = 0, \\ -\frac{\partial H^P}{\partial \alpha} &= \dot{\lambda} - r\lambda \Leftrightarrow \dot{\lambda} = \left(r + \beta\left(p^P\right)\right)\lambda - \tilde{\pi}^P\left(p^I, p^P\right). \end{aligned} \tag{13}$$

And for the incumbent, the standard first-order conditions apply

$$\frac{\partial \Pi^I\left(\alpha, p^I, p^P\right)}{\partial p^i} = 0 \Rightarrow p^I(t) = \phi^I(p^P, \alpha). \tag{14}$$

Condition (13) allows to determine a dynamic version of the platform's best reply (for any

p^I)

$$p^P = \psi^P\left(p^I, \lambda, \alpha\right) : \frac{\partial \tilde{\pi}^P\left(p^I, p^P\right)}{\partial p^P} = -\lambda \frac{1-\alpha}{\alpha} \beta'\left(p^P\right). \tag{15}$$

As a result from (13) and (14), for each (α, λ), one may define a solution $\left(p^P(\alpha, \lambda), p^I(\alpha, \lambda)\right)$ such that

$$p^P(\alpha, \lambda) = \psi^P\left(p^I(\alpha, \lambda), \lambda, \alpha\right) \text{ and } p^I(\alpha, \lambda) = \phi^I\left(p^P(\alpha, \lambda), \alpha\right).$$

where $\phi^I(p^P, \alpha)$ is the best-reply from the static game, defined in Equation (3).

[2] To assess how this solution varies with (α, λ), we need to establish that the price game is supermodular. As a first step, we need to establish that the platform's best-reply defined by the first-order conditions in Equations (13) is increasing in price. This holds true as:

$$\frac{\partial^2 H^P}{\partial p^P \partial p^I} = \alpha \frac{\partial^2 \tilde{\pi}^P}{\partial p^P \partial p^I} \geq 0,$$

and

$$\frac{\partial^2 \Pi^I}{\partial p^I \partial p^P} = \alpha \frac{\partial^2 \tilde{\pi}^I(p^I, p^P)}{\partial p^I \partial p^P} \geq 0.$$

So for all (α, λ), the dynamic price game where the platform objective is $H^P\left(\alpha, p^P, \lambda\right)$ remains smooth supermodular in prices strategies. And prices remain strategic complements for each admissible (α, λ):

$$\frac{\partial \psi^P}{\partial p^I} \geq 0 \ , \ \frac{\partial \phi^I}{\partial p^P} \geq 0.$$

[3] Next to apply Topkis theorem (see Amir, 2005), we compute the cross-derivatives of H^P and Π^I with respect α and λ. If $\lambda \geq 0$, let us define $A = -\alpha$, and $\Lambda = -\lambda$, then we have :

$$H^P\left(A, p^P, \Lambda\right) = -A\tilde{\pi}^P\left(p^I, p^P\right) - f^P - \Lambda(1 + A)\beta\left(p^P\right).$$

thus

$$\begin{aligned}
\frac{\partial^2 H^P}{\partial p^P \partial A} &= -\frac{\partial \tilde{\pi}^P\left(p^I, p^P\right)}{\partial p^P} - \Lambda \beta'\left(p^P\right) = \frac{\Lambda}{A}\beta'\left(p^P\right) \geq 0, \\
\frac{\partial^2 H^P}{\partial p^P \partial \Lambda} &= -\left(1+A\right)\beta'\left(p^P\right) > 0, \\
\frac{\partial \Pi^I}{\partial p^I \partial A} &= -\left(\frac{\partial \tilde{\pi}^I}{\partial p^I} - \frac{\partial \pi^I}{\partial p^I}\right) \geq 0, \\
\frac{\partial \Pi^I}{\partial p^I \partial \Lambda} &= 0.
\end{aligned}$$

Again all functions H^P and Π^I are supermodular in $\left(p^P, p^I, A, \Lambda\right)$, implying they admit increasing differences in (A, Λ) for all prices. Hence applying the Topkis theorem, we have the following *comparative statics result* for the solution $\left(p^P(A,\Lambda), p^I(A,\Lambda)\right)$ are increasing in each (A, Λ). Consequently

$$\begin{aligned}
\frac{\partial p^P(\alpha,\lambda)}{\partial \alpha} &\leq 0 \text{ and } \frac{\partial p^I(\alpha,\lambda)}{\partial \alpha} \leq 0, \\
\frac{\partial p^P(\alpha,\lambda)}{\partial \lambda} &\leq 0 \text{ and } \frac{\partial p^I(\alpha,\lambda)}{\partial \lambda} \leq 0.
\end{aligned}$$

As a consequence, for all α

$$\begin{aligned}
p^P(\alpha,\lambda) &\leq p^P(\alpha, 0) = p_s^P(\alpha), \\
p^I(\alpha,\lambda) &\leq p^I(\alpha, 0) = p_s^I(\alpha).
\end{aligned} \tag{16}$$

and whenever $\lambda \geq 0$ we have the result that for all admissible $\left(\alpha, \lambda, p^I\right)$

$$\psi^P\left(p^I, \lambda, \alpha\right) \leq \psi^P\left(p^I, 0, \alpha\right) = \phi^P\left(p^I\right).$$

[4] Now, we need to prove that, for all $t \geq 0 : \lambda \geq 0$. We proceed in several steps. We first construct the following dynamic system based on the above conditions and we use it to determine the steady state equilibrium. Then (see **[5]**), we qualify the stability of the steady state, which imply $\lambda > 0$ and $\dot{\lambda} < 0$.

$$\begin{aligned}
\dot{\alpha} &:= \mathcal{A}(\alpha,\lambda) = (1-\alpha)\beta\left(p^P(\alpha,\lambda)\right), \\
\dot{\lambda} &:= \mathcal{L}(\alpha,\lambda) = \left(r + \beta\left(p^P(\alpha,\lambda)\right)\right)\lambda - \tilde{\pi}^P(p^I(\alpha,\lambda), p^P(\alpha,\lambda)).
\end{aligned}$$

The steady state equilibrium $(\alpha^{\infty}, \lambda^{\infty})$ is defined as

$$\begin{cases} \alpha^{\infty} = 1 > 0 \\ \lambda^{\infty} = \dfrac{\tilde{\pi}^{P}(p_{\infty}^{I}, p_{\infty}^{P})}{r + \beta\left(p_{\infty}^{P}\right)} > 0 \end{cases} \tag{17}$$

with $p_{\infty}^{I} = p^{I}\left(1, \lambda^{\infty}\right) = p_{s}^{I}\left(1\right)$ and $p_{\infty}^{P} = p^{P}\left(1, \lambda^{\infty}\right) = p_{s}^{P}\left(1\right)$. Using the comparative statics result above, we have:

$$\begin{aligned} \mathcal{A}_{\alpha}'\left(\alpha, \lambda\right) &= -\beta\left(p^{P}\right) + \left(1-\alpha\right)\frac{\partial p^{P}\left(\alpha, \lambda\right)}{\partial \alpha}\beta'\left(p^{P}\right), \\ \mathcal{A}_{\lambda}'\left(\alpha, \lambda\right) &= \left(1-\alpha\right)\frac{\partial p^{P}\left(\alpha, \lambda\right)}{\partial \lambda}\beta'\left(p^{P}\right) \geq 0, \end{aligned}$$

and

$$\mathcal{L}_{\alpha}'\left(\alpha, \lambda\right) = \lambda\beta'\left(p^{P}\right)\frac{\partial p^{P}\left(\alpha, \lambda\right)}{\partial \alpha} - \nabla\tilde{\pi}^{P}(p^{I}, p^{P}) \cdot \left[\frac{\partial p^{I}\left(\alpha, \lambda\right)}{\partial \alpha}, \frac{\partial p^{P}\left(\alpha, \lambda\right)}{\partial \alpha}\right]^{T} \geq 0,$$

$$\mathcal{L}_{\lambda}'\left(1, \lambda^{\infty}\right) = r + \beta\left(p^{P}\right) + \lambda\frac{\partial p^{P}\left(\alpha, \lambda\right)}{\partial \lambda}\beta'\left(p^{P}\right) - \nabla\tilde{\pi}^{P}(p^{I}, p^{P}) \cdot \left[\frac{\partial p^{I}\left(\alpha, \lambda\right)}{\partial \lambda}, \frac{\partial p^{P}\left(\alpha, \lambda\right)}{\partial \lambda}\right]^{T} \geq 0,$$

as at the equilibrium we have

$$\left[\frac{\partial p^{I}\left(\alpha, \lambda\right)}{\partial \lambda}, \frac{\partial p^{P}\left(\alpha, \lambda\right)}{\partial \lambda}\right]^{T} \leq \mathbf{0} \text{ and } \nabla\tilde{\pi}^{P}(p^{I}, p^{P}) = \begin{bmatrix} \frac{\partial\tilde{\pi}^{P}}{\partial p^{I}} \\ -\lambda\frac{1-\alpha}{\alpha}\beta'\left(p^{P}\right) \end{bmatrix} \geq \mathbf{0}.$$

Consequently, the optimal dynamic system exhibits phase lines such that $\dot{\alpha} = 0 \Leftrightarrow \alpha = 1$ and

$$\dot{\lambda} = 0 \Rightarrow \frac{d\lambda}{d\alpha} = -\frac{\mathcal{L}_{\alpha}'\left(\alpha, \lambda\right)}{\mathcal{L}_{\lambda}'\left(\alpha, \lambda\right)} \leq 0.$$

[5] As it is well known, see for instance Theorem 9.5.1. in Léonard and Long (1995), this equilibrium is either unstable or exhibits the saddle-point property. To know that, we study the jacobian of the dynamic system above when $(\alpha, \lambda) = (1, \lambda^{\infty})$. That is

$$J\left(1, \lambda^{\infty}\right) = \begin{bmatrix} -\beta\left(p_{\infty}^{P}\right) & 0 \\ \mathcal{L}_{\alpha}'\left(1, \lambda^{\infty}\right) & \mathcal{L}_{\lambda}'\left(1, \lambda^{\infty}\right) \end{bmatrix}.$$

We can confirm that steady state equilibrium $(1, \lambda^\infty)$ is a saddle-point as we have

$$\mathrm{Tr}\left(J\left(1,\lambda^\infty\right)\right) = \mathcal{L}'_\lambda\left(1,\lambda^\infty\right) - \beta\left(p^P_\infty\right) \geq 0 \text{ and } \left|J\left(1,\lambda^\infty\right)\right| = -\beta\left(p^P_\infty\right)\mathcal{L}'_\lambda\left(1,\lambda^\infty\right) - 0 \leq 0.$$

Consequently, the optimal pricing policy for the platform consists in choosing an appropriate starting value for $\lambda(0) \geq 0$ so that the couple $(\alpha_0, \lambda(0))$ is located on the stable branch of the saddle-point and the equilibrium path $\alpha(t)$ is increasing and monotone with $\lim_{t\to\infty}\alpha(t) = 1$. To go further, one can study the slope of the stable branch, that is associated with the negative eigenvalue of $J(1,\lambda^\infty)$ which is $\nu = -\beta\left(p^P_\infty\right) < 0$. The corresponding eigenvector $\mathbf{e} = (a, l)$ is such that

$$\begin{bmatrix} -\beta\left(p^P_\infty\right) - \nu & 0 \\ \mathcal{L}'_\alpha\left(1,\lambda^\infty\right) & \mathcal{L}'_\lambda\left(1,\lambda^\infty\right) - \nu \end{bmatrix}\mathbf{e} = \begin{bmatrix} 0 & 0 \\ \mathcal{L}'_\alpha\left(1,\lambda^\infty\right) & \mathcal{L}'_\lambda\left(1,\lambda^\infty\right) + \beta\left(p^P_\infty\right) \end{bmatrix}\mathbf{e} = \mathbf{0},$$

or equivalently

$$l = -\frac{\mathcal{L}'_\alpha\left(1,\lambda^\infty\right)}{\mathcal{L}'_\lambda\left(1,\lambda^\infty\right) + \beta\left(p^P_\infty\right)}a \Rightarrow \mathbf{e} = \begin{bmatrix} 1 \\ -\frac{\mathcal{L}'_\alpha(1,\lambda^\infty)}{\mathcal{L}'_\lambda(1,\lambda^\infty)+\beta(p^P_\infty)} \end{bmatrix}.$$

Locally the slope of the stable branch is represented by

$$\frac{d\lambda}{d\alpha} = -\frac{\mathcal{L}'_\alpha\left(1,\lambda^\infty\right)}{\mathcal{L}'_\lambda\left(1,\lambda^\infty\right) + \beta\left(p^P_\infty\right)} < 0.$$

This stable branch locus has a negative slope but lower than the phase line $\dot{\lambda} = 0$. Consequently the (linearized) stable branch is defined by

$$\lambda = \frac{\tilde{\pi}^P(p^I_\infty, p^P_\infty)}{r + \beta\left(p^P_\infty\right)} + (1-\alpha)\frac{\mathcal{L}'_\alpha\left(1,\lambda^\infty\right)}{\mathcal{L}'_\lambda\left(1,\lambda^\infty\right) + \beta\left(p^P_\infty\right)}.$$

Therefore for any α_0, if the value $\lambda(0)$ is chosen such that $\dot{\lambda} > 0$ then it is impossible to converge to the optimal saddle-point path. Therefore, any α_0 lies in the stability domain of the steady state equilibrium $(1, \lambda^\infty)$, the equilibrium path is monotone increasing for the market share α and monotone decreasing for the co-state λ:

$$\dot{\lambda} \leq 0 \quad \text{with} \quad \lambda(0) \in]\lambda^\infty, \bar{\lambda}[\tag{18}$$

with $\bar{\lambda} = \lambda^\infty + \frac{\mathcal{L}'_\alpha(1,\lambda^\infty)}{\mathcal{L}'_\lambda(1,\lambda^\infty)+\beta(p^P_\infty)}$. As a consequence, the platform price cannot be lower than the marginal cost c^P.

To finish, we then denote $p_*^P(t) = p^P(\alpha, \lambda)$ and $p_*^I(t) = p^I(\alpha, \lambda)$, when (α, λ) follows the optimal state and co-state trajectory.

Proof of Lemma 4. Directly from (13) in the proof of Proposition 2 and the value of λ^∞ defined in Equation (17).

Proof of Proposition 3. Directly from (16) in the proof of Proposition 2.

Proof of Proposition 4. From Proposition 3, for any α we have

$$p^P(\alpha, \lambda) \leq p_s^P(\alpha) \Rightarrow \beta\left(p^P(\alpha, \lambda)\right) \geq \beta\left(p_s^P(\alpha)\right).$$

So for any share α

$$\dot{\alpha}_* = (1-\alpha)\,\beta\left(p^P(\alpha, \lambda)\right) \geq \dot{\alpha}_m = (1-\alpha)\,\beta\left(p_s^P(\alpha)\right) \geq 0.$$

Consequently at any time $\alpha_*(t) \geq \alpha_m(t)$.

Proof of Lemma 5. For any price p^I, differentating Equation (13) with respect to time t implies

$$\dot{\psi}^P \cdot \left(\frac{\partial^2 \tilde{\pi}^P\left(p^I, p^P\right)}{\partial\left(p^P\right)^2} + \lambda\frac{1-\alpha}{\alpha}\beta''(p^P)\right) = \left(\dot{\lambda}\frac{1-\alpha}{\alpha} - \frac{\lambda}{\alpha^2}\right)\beta'\left(p^P\right),$$

with $p^P = \psi^P(p^I, \alpha, \lambda)$. From Proposition 2, the RHS is negative and from the concavity of the Hamiltonian H^P, we have

$$\frac{\partial^2 \tilde{\pi}^P\left(p^I, p^P\right)}{\partial\left(p^P\right)^2} + \lambda\frac{1-\alpha}{\alpha}\beta''(p^P) < 0.$$

As a result, $\dot{\psi}^P > 0$. Convergence is straightforward as $\psi^P(p^I, 1, \lambda) = \phi^P(p^I)$.

Proof of Corollary 2. First we see that

$$p(t) = p^P(\alpha_*(t), \lambda(t)) \Rightarrow \dot{p}(t) = \frac{\partial p^P(\alpha, \lambda)}{\partial \alpha}\dot{\alpha}_*(t) + \frac{\partial p^P(\alpha, \lambda)}{\partial \lambda}\dot{\lambda}(t),$$

Then as $\frac{\partial p^P(\alpha,\lambda)}{\partial \alpha} \leq 0$, $\frac{\partial p^P(\alpha,\lambda)}{\partial \lambda} \leq 0$, $\dot{\lambda}(t) \leq 0$ and $\dot{\alpha}_*(t) \geq 0$, so $\dot{p}(t)$ is not necessarily negative.

Proofs of Lemma 6 and Corollary 4. On the basis of the proof elements developed earlier in this appendix, one can write the new Hamiltonian of the platform's problem:

$$H^P\left(\alpha, a, p^P, \lambda, \mu\right) = \alpha\tilde{\pi}^P(p^I, p^P) - \alpha\tilde{x}^P(p^I, p^P)R\left(Y\right) - f^P + \lambda\left(1-\alpha\right)\beta\left(p^P\right) + \mu\left(1-a\right)\gamma\left(Y\right).$$

The Pontryagin's maximum principle implies for the platform that $\forall t \geq 0$,

$$\begin{aligned}
\frac{\partial H^P}{\partial p^P} &= 0 \\
&\Leftrightarrow \frac{\partial \tilde{\pi}^P\left(p^I, p^P\right)}{\partial p^P} - \frac{\alpha}{a}\tilde{x}^P R'\left(Y\right)\frac{\partial \tilde{x}^P}{\partial p^P} = -\lambda\frac{1-\alpha}{\alpha}\beta'\left(p^P\right) - \mu\frac{1-a}{a}\gamma'\left(R\left(Y\right)\right)R'\left(Y\right)\frac{\partial \tilde{x}^P}{\partial p^P}, \\
-\frac{\partial H^P}{\partial \alpha} &= \dot{\lambda} - r\lambda \Leftrightarrow \dot{\lambda} = \left(r + \beta\left(p^P\right)\right)\lambda - \tilde{\pi}^P(p^I, p^P) + \\
&\qquad + \tilde{x}^P\left\{R\left(Y\right) + \left[\alpha - \mu\left(1-a\right)\gamma'\left(R(Y)\right)\right]\frac{R'\left(Y\right)}{a}\right\}, \\
-\frac{\partial H^P}{\partial a} &= \dot{\mu} - r\mu \Leftrightarrow \dot{\mu} = \left(r + \gamma\left(R(Y)\right) + \frac{1-a}{a}\gamma'\left(R\left(Y\right)\right)R'\left(Y\right)Y\right)\mu + \left(\frac{\alpha}{a}\tilde{x}^P(p^I, p^P)\right)^2 R'\left(Y\right).
\end{aligned}$$

This allows to determine the new dynamic version of the platform's best reply (for any p^I) such that for $p^P = \hat{\psi}^P\left(p^I\right)$:

$$\frac{\partial \tilde{\pi}^P\left(p^I, p^P\right)}{\partial p^P} = -\lambda\frac{1-\alpha}{\alpha}\beta'\left(p^P\right) - \mu\frac{1-a}{a}\gamma'\left(R\left(Y\right)\right)R'\left(Y\right)\frac{\partial \tilde{x}^P}{\partial p^P} + \frac{\alpha}{a}\tilde{x}^P R'\left(Y\right)\frac{\partial \tilde{x}^P}{\partial p^P}.$$

Now the steady state is given by

$$\hat{\alpha}^\infty = \hat{a}^\infty = 1 \Rightarrow \frac{\partial \tilde{\pi}^P\left(\hat{p}^I_\infty, \hat{p}^P_\infty\right)}{\partial p^P} = R'\left(Y_\infty\right)\frac{\partial \tilde{x}^P}{\partial p^P} < 0,$$

where $Y_\infty = \tilde{x}^P(\hat{p}^I_\infty, \hat{p}^P_\infty)$ with $\hat{p}^P_\infty > p^P_\infty = p^{*P}\left(1\right)$ and $\hat{p}^I_\infty > p^I_\infty = p^{*I}\left(1\right)$.. As a result

$$\begin{aligned}
\lambda^\infty &= \frac{\tilde{\pi}^P\left(\hat{p}^I_\infty, \hat{p}^P_\infty\right) - \tilde{x}^P\left(\hat{p}^I_\infty, \hat{p}^P_\infty\right)\left[R\left(Y_\infty\right) + R'\left(Y_\infty\right)\right]}{r + \beta\left(\hat{p}^P_\infty\right)}, \\
\mu^\infty &= \frac{\tilde{x}^P\left(\hat{p}^I_\infty, \hat{p}^P_\infty\right)}{r + \gamma\left(R\left(Y_\infty\right)\right)}R'\left(Y_\infty\right) > 0.
\end{aligned}$$

We do not study convergence to the steady state and it may not be ensured (limit cycles are possible). But, if convergence occurs, in the long run both affiliations have a positive shadow value, implying higher prices for consumers at the steady state.

B Illustration: the Singh and Vives model

In this appendix, we develop an illustrative example based on Singh and Vives (1984).

Utility function The utility of a consumer j is given by

$$U_j(x^I, x^P) = a(x^I + x^P) - b\frac{(x^I)^2 + (x^P)^2}{2} - dx^I x^P, \text{with} \;\; b > d > 0.$$

Demand functions A non-affiliated consumer j chooses a consumption level to $\max_{x^I} U_j(x^I, 0) - p^I x^I$. From that, we can derive the demand function:

$$x^I(p^I) = \frac{a - p^I}{b}.$$

The demands from an affiliated consumer j are found by solving:

$$\max_{x^I, x^P} U_j(x^I, x^P) - p^I x^I - p^P x^P,$$

and are given by:

$$\tilde{x}^I(p^I, p^P) = \frac{a(b-d) - bp^I + dp^P}{b^2 - d^2} \quad \text{and} \quad \tilde{x}^P(p^I, p^P) = \frac{a(b-d) - bp^P + dp^I}{b^2 - d^2}.$$

These demand functions satisfy Assumption 1 as[20]

$$\begin{aligned}
\frac{\partial \tilde{x}^I(p^I, p^P)}{\partial p^P} &= \frac{\partial \tilde{x}^P(p^I, p^P)}{\partial p^I} = \frac{d}{b^2 - d^2} > 0, \\
\tilde{x}^I(p^I) - x^I(p^I, p^P) &= \frac{ad\,(b-d) + d^2 p^I}{b\,(b^2 - d^2)} - \frac{dp^P}{b^2 - d^2} \geq 0 \text{ for } p^P \leq p^I, \\
\frac{\partial \tilde{x}^I(p^I, p^P)}{\partial p^I} - \frac{\partial x^I(p^I)}{\partial p^I} &= \frac{-b}{b^2 - d^2} + \frac{1}{b} = -\frac{d^2}{b\,(b^2 - d^2)} < 0.
\end{aligned}$$

Best reply functions Assuming zero unit costs for firms, simple calculation show that best replies entail

$$\begin{aligned}
\phi^P(p^I) &= \frac{d}{2b} p^I + \frac{a\,(b-d)}{2b}, \\
\phi^I(p^P, \alpha) &= \frac{1}{2} \frac{bd}{b^2 - (1-\alpha)\,d^2}\, \alpha p^P - \frac{1}{2} \frac{ad\,(b-d)\,\alpha - a\left(b^2 - d^2\right)}{b^2 - (1-\alpha)\,d^2}.
\end{aligned}$$

[20]There is an upper bound in the set of admissible prices p^P.

So we have we can verify Lemma 1

$$\frac{\partial \phi^I(p^P,\alpha)}{\partial p^I} = \frac{1}{2}\frac{bd}{b^2-(1-\alpha)\,d^2}\alpha \geq 0 \text{ and } \frac{\partial \phi^I(p^P,\alpha)}{\partial \alpha} = -\frac{bd\left(b^2-d^2\right)}{2\left(b^2-(1-\alpha)\,d^2\right)}\left(a-p^P\right) \leq 0.$$

Static model: equilibrium prices The static price equilibrium is then

$$p_s^P(\alpha) = a\frac{(2b-d)\left(b^2-d^2\right)+d^2(b-d)\,\alpha}{b\left(4\left(b^2-d^2\right)+3\alpha d^2\right)} \quad\text{and}\quad p_s^I(\alpha) = a\frac{2(b^2-d^2)-d\,(b-d)\,\alpha}{4\left(b^2-d^2\right)+3\alpha d^2},$$

so

$$\frac{\partial p_s^P(\alpha)}{\partial \alpha} = -\frac{ad^2\,(d+2\,b)\left(b^2-d^2\right)}{\left(4\left(b^2-d^2\right)+3\alpha d^2\right)^2} < 0 \quad\text{and}\quad \frac{\partial p_s^I(\alpha)}{\partial \alpha} = -2\frac{ad\,(d+2\,b)\left(b^2-d^2\right)}{\left(4\left(b^2-d^2\right)+3\alpha d^2\right)^2} < 0.$$

This illustrates Lemma 2.

Static game: equilibrium profits The profits are given by

$$\begin{aligned}
\pi^I &= \frac{(b-d)\left(b^2-(1-\alpha)d^2\right)(a\alpha d-2a(b+d))^2}{b(b+d)\left(4b^2-4d^2+3\alpha d^2\right)^2},\\
\pi^P &= \frac{a^2\alpha(b-d)\left(2b^2+bd-(1-\alpha)d^2\right)^2}{b(b+d)\left(4\left(b^2-d^2\right)+3\alpha d^2\right)^2}.
\end{aligned}$$

π^I is decreasing in α, π^P is increasing in α and it is concave if (sufficient condition) $d \leq 2/3b$.

Dynamic affiliation For all $\left(p^P,p^I\right)$ we have the following net utilities of a non-affiliated and an affiliated consumer j

$$\begin{aligned}
U_j^n &= U_j(x^I,0)-p^Ix^I = \frac{1}{2b}(a-p^I)^2,\\
U_j^a &= U_j(\tilde{x}^I,\tilde{x}^P)-p^I\tilde{x}^I-p^P\tilde{x}^P,\\
&= \frac{b}{2\left(b^2-d^2\right)}\left(\left(p^P\right)^2+\left(p^I\right)^2\right)-\frac{b}{2\left(b^2-d^2\right)}p^Ip^P-\frac{a}{b+d}\left(p^P+p^I\right)+\frac{a^2}{b+d}.
\end{aligned}$$

The net gain from opting in is

$$\Delta U = U_a^* - U_n^* = \frac{\left(b\left(a-p^P\right)-d(a-p^I)\right)^2}{2b\left(b^2-d^2\right)} > 0,$$

which decreases and is convex in p^p.

Dynamic game Let us now assume a dynamic affiliation motion such that $\beta\left(p^P\right) = \zeta - \delta p^P \geq 0$. Then the (dynamic) best reply to α, λ, p^I for the platform is $\psi^P\left(p^I, \lambda, \alpha\right)$:

$$\psi^P\left(p^I, \lambda, \alpha\right) = \phi^P(p^I) - \delta\frac{\left(b^2-d^2\right)}{2b}\lambda\frac{1-\alpha}{\alpha}.$$

As a result we derive the Nash-equilibrium for all (α, λ) such that $p^P(\alpha,\lambda) = \psi^P\left(p^I(\alpha,\lambda), \lambda, \alpha\right)$ and $p^I(\alpha,\lambda) = \phi^I\left(p^P(\alpha,\lambda), \alpha\right)$ that is

$$\begin{aligned}
p_*^I(\alpha,\lambda) &= a(b-d)\frac{2(b+d)-\alpha d}{4(b^2-d^2)+3\alpha d^2} - \frac{d\delta\left(b^2-d^2\right)}{4(b^2-d^2)+3\alpha d^2}\lambda(1-\alpha), \\
&= p_s^I(\alpha) - \frac{d\delta\left(b^2-d^2\right)}{4(b^2-d^2)+3\alpha d^2}\lambda(1-\alpha). \\
p_*^P(\alpha,\lambda) &= a(b-d)\frac{(2b-d)(b+d)+\alpha\, d^2}{b(4(b^2-d^2)+3\alpha d^2)} - 2\,\frac{\delta\left(b^2-d^2\right)\left((b^2-d^2)+\alpha d^2\right)}{\alpha b(4(b^2-d^2)+3\alpha d^2)}\lambda(1-\alpha), \\
&= p_s^P(\alpha) - 2\,\frac{\delta\left(b^2-d^2\right)\left((b^2-d^2)+\alpha d^2\right)}{\alpha b(4(b^2-d^2)+3\alpha d^2)}\lambda(1-\alpha).
\end{aligned}$$

So he underlying dynamic system in (α, λ) writes

$$\begin{aligned}
\dot{\alpha} &= (1-\alpha)\left(\zeta - \delta p^P(\alpha,\lambda)\right), \\
\dot{\lambda} &= \left(r+\zeta-\delta p^P(\alpha,\lambda)\right)\lambda - p^P(\alpha,\lambda)\,\tilde{x}^P(p^I(\alpha,\lambda), p^P(\alpha,\lambda)).
\end{aligned}$$

The steady-state is equal to

$$\alpha^\infty = 1 \text{ and } \lambda^\infty = \frac{a^2 b(b-d)}{(d+b)(2b-d)((2\,b-d)(r+\zeta)-\delta a(b-d))},$$

and terminal prices are $p_s^P(1) = a\frac{b-d}{2b-d}$ and $p_s^I(1) = a\frac{b-d}{2b-d}$.