Price Taking as the Asymptotic Limit of Strategic Behavior

Leonidas C. Koutsougeras [a,b]

[a] *School of Social Sciences, University of Manchester, Arthur Lewis 3.016, Oxford Road, Manchester M13 9PL, UK.*

[b] *CORE, Université Catholique de Louvain, 34 Voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium.*

1 Introduction

It is widely accepted among economists that in a system of markets where individual participants are small relative to the market size, individuals have a negligible effect on the determination of market outcomes, so they may be thought of as exhibiting a 'price taking' behavior. Of course, in order to make sense of this statement one has to attach a meaning to 'a small individual relative to market size'. In this way it would be possible to distinguish when price taking is a reasonable assumption and when it is not. The significance of the price taking hypothesis in economics calls for a formal clarification of this point -a 'theory of competition' so to speak.

One of the tools of economic theory to this effect is the asymptotic study of equilibrium outcomes of finite economies, when the number of individuals increases without limit. The idea is that if we can identify conditions under which equilibrium outcomes of finite economies converge asymptotically (in some sense) to Walrasian ones, then we would have a context where individuals have negligible effect on market outcomes and hence when 'price taking' can be justified as a reasonable hypothesis. An alternative way to view the asymptotic study is as a link between 'large finite' and 'atomless' economies. In atomless economies 'negligibility' is built in the non atomicity of the measure of the space of agents. If a large finite economy is to be thought as a reasonable substitute of the idealized continuum model, it should be the case that equilibrium outcomes of a large finite economy are close to those of the atomless limit, i.e., the equilibrium outcomes of the former should asymptotically converge (in some sense) to those of the latter as the number of individuals increases.

Asymptotic studies have been performed for a variety of equilibrium notions which have been developed in finite economies. For the most part asymptotic studies focus on the limits of the core [1] and Nash equilibria, mainly because those notions are associated with the traditional theories of Edgeworth and Cournot, which are prevalent in economic theory.

In this paper we study the asymptotic limits of Nash equilibria of strategic market games. This issue has been addressed by several authors, Dubey and Shubik (1978), Mas-Colell (1982), Peck and Shell (1989), Sahi and Yao (1989), Amir et. al. (1990) among many others, albeit in the fragile context of sequences of economies obtained through replication. [2] Besides the particularity of this type of sequences (finite number of types of individual characteristics), the above results are shown only for 'type symmetric' Nash equilibria. Note that in replica sequences type symmetry is a property of the core (known as 'equal treatment'), but not of Nash equilibria.

By contrast, our results apply to more general sequences of economies with characteristics drawn from compact sets and do not depend on type symmetry. One of our results provides also a rate of convergence. In this way we address the issue of asymptotic convergence of Nash equilibria, at the same level of generality as some known core convergence results. Our approach is based on the idea in Koutsougeras (2007) of measuring individuals' departure from price taking, via the wedge between the hyperplanes defined by the price vector and the supporting hyperplane of the indifference surface through the equilibrium bundle. We then demonstrate that under suitable assumptions on the distribution of individual characteristics [3] this wedge becomes arbitrarily small as the number of individuals converges to infinity.

[1] See Anderson (1992) for a survey of core equivalence results and references.

[2] In some related papers but somewhat distinct in scope, Peck and K. Shell (1990) features an asymptotic exercise where the number of agents remains finite but the volume of trade increases without limit, while Postlewaite and Schmeidler (1981) shows that Nash equilibria are Walrasian for a properly defined 'nearby economy'.

[3] Remarkably these assumptions are the same as in the case of core convergence.

2 The model

Let H be a finite set of agents. There are L commodity types in the economy and the consumption set of each agent is identified with $\Re_+^L$. Each individual $h \in H$ is characterized by a preference relation $\succeq_h \subset \Re_+^L \times \Re_+^L$ and an initial endowment $e_h \in \Re_+^L \setminus \{0\}$. We use the following assumption:

ASSUMPTION 1 *Preferences are continuous, convex and strictly monotone.*

Denote by $\mathcal{P}_{cm}$ the set of preferences that satisfy (1) endowed with the topology of closed convergence. Let $T \subset \mathcal{P}_{cm} \times \Re_+^L$. An economy is defined as a mapping $\mathcal{E} : H \to T$.

We now turn to describe a strategic market game, which proposes an explicit model of how exchange in the economy takes place.

2.1 Trade using inside money

We will develop our results for the strategic market game version appearing in Postlewaite and Schmeidler (1978) and in Peck et. al. (1992) which is described below.

Trade in the economy is organized via a system of trading posts where individuals offer commodities for sale and place bids for purchases of commodities. Bids are placed in terms of a unit of account. The strategy set of each agent is $S_h = \{(b_h, q_h) \in \Re_+^{2L} : q_h^i \leq e_h^i,\ i = 1, 2, \ldots, L\}$. Given a strategy profile $(b, q) \in \Pi_{h\in H} S_h$ let $B^i = \sum_{h\in H} b_h^i$ and $Q^i = \sum_{h\in H} q_h^i$ denote aggregate bids and offers for each $i = 1, 2, \ldots, L$. Also for each agent h denote $B_{-h}^i = \sum_{k\neq h} b_k^i$, $Q_{-h}^i = \sum_{k\neq h} q_k^i$. For a given a strategy profile, the consumption of consumer $h \in H$ is determined by $x_h = e_h + z_h(b, q)$, where for $i = 1, 2, \ldots, L$:

$$z_h^i(b, q) = \begin{cases} \frac{b_h^i}{B^i} Q^i - q_h^i & \text{if } \sum_{i=1}^L \frac{B^i}{Q^i} q_h^i \geq \sum_{i=1}^L b_h^i \\ -q_h^i & \text{otherwise} \end{cases} \tag{1}$$

and it is postulated that whenever the term $0/0$ appears in the expressions above it is defined to equal zero. When $B^i Q^i \neq 0$ the fraction $\pi^i(b, q) = \frac{B^i}{Q^i}$

has a natural interpretation as the (average) market clearing 'price'. The relation $\sum_{i=1}^{L} \pi^i(b,q)q_h^i \geq \sum_{i=1}^{L} b_h^i$ is a 'bookkeeping' restriction which ensures that units of account remain at zero net supply (inside money). The interpretation of this allocation mechanism is that commodities (money) is distributed among non bankrupt consumers in proportion to their bids (offers), while the purchases of bankrupt consumers are confiscated.

An equilibrium is defined as a strategy profile $(b,q) \in \prod_{h \in H} S_h$ that forms a Nash equilibrium in the ensuing game with strategic outcome function given by (1). Let $\mathbf{N}(\mathcal{E}) \subset \prod_{h \in H} S_h$ denote the set of Nash equilibrium strategy profiles of the strategic market game and $\mathcal{N}(\mathcal{E}) \subset \Re_+^{LH}$ the set of consumption allocations corresponding to the elements of $\mathbf{N}(\mathcal{E})$.

The following notation and familiar facts will be useful in the sequel. Fix $(b_{-h}, q_{-h}) \in \prod_{k \neq h} S_k$ and let[4] $g_h(y) = \sum_{i=1}^{L} \frac{B_{-h}^i (y^i - e_h^i)}{Q_{-h}^i + e_h^i - y^i}$. The set of allocations which an individual $h \in H$ can achieve via the strategic outcome function is given by the convex set

$$c_h = \{y \in \Re_+^L : g_h(y) \leq 0, y \leq Q_{-h} + e_h\}$$

i.e., $(b_h, q_h) \in S_h \Rightarrow e_h + z_h(b,q) \in c_h$. Conversely, $x_h \in c_h \Rightarrow \exists (b_h, q_h) \in S_h \ s.t. \ x_h = e_h + z_h(b,q)$. Thus, due to the bankruptcy rule, at an equilibrium with nonzero bids and offers we have: $\bar{x} \in \mathcal{N}(\mathcal{E})$ if and only if:

$$\begin{aligned} &(i)\ \bar{x} = x(\bar{b}, \bar{q}),\ \text{for some } (\bar{b}, \bar{q}) \in \prod_{h \in H} S_h \\ &(ii)\ \forall h \in H,\ c_h \cap \{y \in \Re_+^L : y \succ_h \bar{x}_h\} = \emptyset \end{aligned} \tag{2}$$

We say that $\bar{x} \in \mathcal{N}(\mathcal{E})$ is fully active if for the corresponding $(\bar{b}, \bar{q}) \in \mathbf{N}(\mathcal{E})$ we have $\pi(\bar{b}, \bar{q}) \gg 0$, i.e., there is trade in all commodities. In the sequel we will focus on such equilibria.[5]

2.2 Strategic vs price taking behavior

Let us fix a fully active $\bar{x} \in \mathcal{N}(\mathcal{E})$ corresponding to a strategy profile $(\bar{b}, \bar{q}) \in \mathbf{N}(\mathcal{E})$. Consider one $h \in H$ and denote $\bar{z}_h = \bar{x}_h - e_h$.

[4] In order to save on notation we omit the dependency on (b_{-h}, q_{-h}). In the results the values of those variables will be fixed so no confusion should arise.
[5] Alternatively we could consider the subset of commodities L' for which there is active trade.

The monotonicity of preferences implies that $g_h(\bar{x}_h) = 0$, i.e., $\bar{x}_h$ lies on the boundary of the convex set c_h, which is C^2. Since preferences are also convex, by the separating hyperplane theorem there is a $p_h \in \Re^L_+$, specifically $p_h = Dg_h(\hat{x}_h)$, where $Dg_h(\cdot)$ denotes the gradient of $g_h(\cdot)$, such that

$$w \succeq_h \bar{x}_h \Rightarrow p_h w \geq p_h \bar{x}_h \text{ and } w \in c_h \Rightarrow p_h w \leq p_h \bar{x}_h \tag{3}$$

Using the definition of c_h we have

$$p_h = Dg_h(\bar{x}_h) = \left(\frac{\bar{B}^i_{-h} \bar{Q}^i_{-h}}{(\bar{Q}^i_{-h} - \bar{z}^i_h)^2} \right)^L_{i=1} = \left(\pi^i(\bar{b}, \bar{q}) \frac{\bar{Q}^i_{-h}}{(\bar{Q}^i_{-h} - \bar{z}^i_h)} \right)^L_{i=1} \tag{4}$$

Now observe that if for some $\lambda_h > 0$, $p_h = \lambda_h \pi(\bar{b}, \bar{q})$ then the behavior of such an individual would be identical to price taking at the market clearing prices $\pi(\bar{b}, \bar{q})$. To see this notice that because $\pi(\bar{b}, \bar{q}) \gg 0$ ($\bar{x}$ is active) there is a cheaper point, i.e., $w \in \Re^L_+$ with $\pi(\bar{b}, \bar{q})w < \pi(\bar{b}, \bar{q})\bar{x}_h = \pi(\bar{b}, \bar{q})e_h$. Since furthermore preferences are continuous and convex, the first part of (3) implies $y \succ_h \bar{x}_h \Rightarrow \pi(\bar{b}, \bar{q})y > \pi(\bar{b}, \bar{q})e_h$. Finally, $\pi(\bar{b}, \bar{q})\bar{x}_h = \pi(\bar{b}, \bar{q})e_h$.

Therefore, the measurement

$$\delta_h(\bar{x}) = \max \left\{ \left| \frac{p^i_h}{p^j_h} \cdot \frac{\pi^j(\bar{b}, \bar{q})}{\pi^i(\bar{b}, \bar{q})} - 1 \right| : i, j = 1, 2, \ldots, L \right\} \tag{5}$$

serves as an indicator of 'how far' the strategic behavior of individual h falls from price taking. [6] Clearly, for each agent h we have $\delta_h(\bar{x}) \geq 0$ and $\bar{x}$ is Walrasian if (and only if) $\delta_h(\bar{x}) = 0$ for each agent h. Therefore, a sequence of market game price-allocation pairs tends to become a price taking one, if (and only if) the above indicator tends to zero (in an appropriate sense) for all individuals.

We are ready now to proceed with the results of this paper.

3 Results

For the results that follow we consider a sequence $\{\mathcal{E}_n\}_{n \in N}$ of economies $\mathcal{E}_n : H_n \to \mathcal{P}_{cm} \times [0, r]^L$, where $\#H_n \to \infty$, $\lim \frac{1}{\#H_n} \sum_{h \in H_n} e_h \gg 0$ and

[6] In the case of C^2 preferences, the indicator $\delta_h(\cdot)$ coincides with $\gamma_h(\cdot)$ in Koutsougeras (2007).

associated $x_n \in \mathcal{N}(\mathcal{E}_n)$, for each $n \in N$ which are fully active. Let $(b_n, q_n) \in \mathbf{N}(\mathcal{E}_n)$, be the corresponding strategies and $z_{n,h} = x_{n,h} - e_h$ the corresponding net trades for each $h \in H$.

The following result is shown in Koutsougeras (2007) and its proof applies unchanged here.

THEOREM 1 *For each* $\epsilon > 0$, *there is an* $n_\epsilon \in N$ *so that for all* $n > n_\epsilon$

$$\frac{1}{\#H_n} \cdot \#\{h \in H_n : \delta_h(x_n) > \epsilon\} < \epsilon$$

or equivalently

$$\frac{1}{\#H_n} \cdot \#\{h \in H_n : \delta_h(x_n) > \epsilon\} \to 0$$

The above theorem asserts some kind of convergence (in measure) for our indicator. Its strength is that it requires no assumptions, so it applies to all sequences of active Nash equilibria. On the other hand it hardly fits the bill: we still need to show some convergence of the Nash equilibrium allocations themselves. Furthermore, we need to ensure that 'most' of the commodities are consumed by 'most' of the individuals who exhibit 'almost' a price taking behavior, as the above theorem asserts. We proceed with two lemmas which will be useful to us in pursuing this end.

LEMMA 1 *Define* $A(k) = \left\{h \in H_n :\mid z^i_{n,h} \mid \leq kr \text{ for } i = 1, 2, \ldots, L\right\}$, *where* $k \geq L - 1$. *Then* $\#A(k) \geq (1 - \frac{L}{k+1})\#H_n$.

Proof :

Define $T_i(k) = \left\{h \in H_n :\mid z^i_{n,h} \mid > k \cdot r\right\}$, for $i = 1, 2, \ldots, L$. Notice that $T_i(k) = \left\{h \in H_n : z^i_{n,h} > k \cdot r\right\} \cup \left\{h \in H_n : z^i_{n,h} < -k \cdot r\right\}$. However, the second term is empty so $T_i(k) = \left\{h \in H_n : z^i_{n,h} > kr\right\}$.

Also notice that $\#A(k) = \#H_n - \#\left(\cup_{i=1}^L T_i(k)\right)$.

From the definition of $T_i(k)$ it follows that:

$$\#T_i(k) \cdot k \cdot r < \sum_{h \in T_i(k)} z^i_{n,h}$$

$$\begin{aligned}
&= Q_n^i \cdot \sum_{h \in T_i(k)} \frac{b_{n,h}^i}{B_n^i} - \sum_{h \in T_i(k)} q_{n,h}^i \\
&\le Q_n^i - \sum_{h \in T_i(k)} q_{n,h}^i \\
&= \sum_{h \neq T_i(k)} q_{n,h}^i \\
&\le (\#H_n - \#T_i(k)) \cdot r
\end{aligned}$$

Therefore, we conclude that $\#T_i(k) < \#H_n \cdot \frac{1}{k+1}$.

It follows that $\#\left(\cup_{i=1}^L T_i(k)\right) \le \sum_{i=1}^L T_i(k) < \#H_n \cdot \frac{L}{k+1}$.

Hence, $\#A(k) = \#H_n - \#\left(\cup_{i=1}^L T_i(k)\right) \ge \#H_n \cdot \left(1 - \frac{L}{k+1}\right)$ □

LEMMA 2 *Suppose that* $\lim_{n\to\infty} \frac{1}{\#H_n} \sum_{h \in H_n} e_h 1_L \ge a 1_L \gg 0$. *There is a subsequence (still indexed by* n*), and* $\epsilon > 0$ *so that, for each* $i = 1, 2, \ldots, L$ *we have:* $\#\{h \in H_n : x_{n,h}^i \ge \epsilon\} \ge \#H_n \epsilon$.

Proof:

Suppose not. Then for each $\epsilon > 0$ we have for n large enough there is $i = 1, 2, \ldots, L$ so that $\frac{1}{\#H_n}\#\{h \in H_n : x_{n,h}^i \ge \epsilon\} < \epsilon$. By passing to a subsequence if necessary it can be assumed that for some i we have $\frac{1}{\#H_n}\#\{h \in H_n : x_{n,h}^i \ge \epsilon\} < \epsilon$ for n large enough.

For each $M > 0$ consider a truncation $\{x_n^{i,M}\}_{M \in N}$ of the original sequence:

$$x_{n,h}^{i,M} = \begin{cases} x_{n,h}^i & \text{if } x_{n,h}^i < M \\ M & \text{otherwise} \end{cases} \tag{6}$$

This sequence is non decreasing, $x_n^{i,M} \le x_n^i \ \forall M \in N$ and $x_n^{i,M} \to x_n^i$ as $M \to \infty$.

Given $0 < \epsilon < M$ we have that:

$$\begin{aligned}
\sum_{h \in H_n} x_{n,h}^{i,M} = & \sum_{\{h \in H_n : x_{n,h}^{i,M} \ge \epsilon\}} x_{n,h}^{i,M} + \sum_{\{h \in H_n : x_{n,h}^{i,M} < \epsilon\}} x_{n,h}^{i,M} \\
& < M\#\{h \in H_n : x_{n,h}^{i,M} \ge \epsilon\} + \epsilon\#\{h \in H_n : x_{n,h}^{i,M} < \epsilon\} \\
& = M\#\{h \in H_n : x_{n,h}^i \ge \epsilon\} + \epsilon\#\{h \in H_n : x_{n,h}^i < \epsilon\}
\end{aligned}$$

$< M\epsilon + \epsilon$

Therefore, for each M we have $\lim_{n\to\infty} \frac{1}{\#H_n} \sum_{h\in H_n} x^{i,M}_{n,h} = 0$.

Fix $0 < \delta < a$. We have that:

$$\forall M, \exists n_M \in N \ s.t. \ \frac{1}{\#H_n} \sum_{h\in H_n} x^{i,M}_{n,h} < \delta, \forall n \geq n_M$$

In particular, $\frac{1}{\#H_{n_M}} \sum_{h\in H_{n_M}} x^{i,M}_{n_M,h} < \delta$, for all M. Since $x^{i,M}_n \to x^i_n$ we have that for each index n_M, $\frac{1}{\#H_{n_M}} \sum_{h\in H_{n_M}} x^i_{n_M,h} \leq \delta$.

But $\lim_{n\to\infty} \frac{1}{\#H_n} \sum_{h\in H_n} x^i_{n,h} = \lim_{n\to\infty} \frac{1}{\#H_n} \sum_{h\in H_n} e^i_h \geq a > \delta$, which implies that for n_M large enough $\frac{1}{\#H_{n_M}} \sum_{h\in H_{n_M}} x^i_{n_M,h} > \delta$, contradicting the preceding statement. This contradiction establishes the claim of the lemma. □

We now turn to develop an asymptotic convergence theorem, by introducing appropriate assumptions on the distribution of characteristics along a sequence of economies. In particular, consider a sequence of economies $\mathcal{E}_n : H_n \to T$ where $T \subset \mathcal{P}_{cm} \times [0, r]^L$ is compact. For such sequences the set of Nash equilibrium allocations is uniformly bounded as the following result shows.

PROPOSITION 1 *Let* $\{\mathcal{E}_n\}_{n\in N}$ *be a sequence of economies,* $\mathcal{E}_n : H_n \to T$ *where* $\#H_n \to \infty$ *and let* $x_n \in \mathcal{N}(\mathcal{E}_n)$*, for each* $n \in N$ *be fully active. There is* $B \subset \Re^L_+$*, which is bounded and depends only on* T*, such that for all* $n \in N$ $x_{n,h} \in B$ *for each* $h \in H_n$*, i.e., the set of Nash equilibrium allocations remains uniformly bounded along a sequence of economies with characteristics drawn from* T*.*

Proof:

Step I Let $\pi_n = \pi(b_n, q_n)$ and normalize prices so that $\sum_{i=1}^L \pi^i_n = 1$.

Suppose that $sup\{x^j_{n,h} : h \in H_n\} \to \infty$ for some $j = 1, 2, \ldots, L$. Then it must be $sup\{\frac{b^j_{n,h}}{\pi^j_n} - q^j_{n,h} : h \in H_n\} \to \infty$. It follows that $\pi^j_n \to 0$. Hence, there must be $\pi^i_n > 1/L$ for some $i \neq j$ along a subsequence, so $\pi^i_n/\pi^j_n \to \infty$.

Step II By lemma (2), passing to a subsequence if necessary, we may assume that for some $1 > \epsilon > 0$,

$$\#\{h \in H_n : x^i_{n,h} \geq \epsilon\} \geq \#H_n\epsilon \tag{7}$$

Also by lemma 1, setting $k \geq 2L\epsilon^{-1} - 1$, we have that for all $n \in N$,

$$\#\left\{h \in H_n :\mid z^i_{n,h} \mid\leq (2L\epsilon^{-1} - 1)r \ \forall i = 1, 2, \ldots, L\right\} > \#H_n(1 - \frac{\epsilon}{2}) \tag{8}$$

Step III We now show the following claim: for some subsequence (still indexed by n) there exists $M > 0$ so that:

$$\#\{h \in H_n : \frac{p^i_h}{p^j_h} \leq M\} > \#H_n\frac{\epsilon}{2} \tag{9}$$

Suppose not. Then for every $M > 0$ we have $\#\{h \in H_n : \frac{p^i_h}{p^j_h} \leq M\} \leq \#H_n\frac{\epsilon}{2}$ or equivalently $\#\{h \in H_n : \frac{p^i_h}{p^j_h} > M\} > \#H_n(1 - \frac{\epsilon}{2})$. In conclusion we have for every $M > 0$

$$\#\{h \in H_n : \frac{p^j_h}{p^i_h} < M^{-1}\} > \#H_n(1 - \frac{\epsilon}{2}) \tag{10}$$

In this case, (10) along with (7) and (8), imply that for each $n \in N$ there is $h_n \in H_n$ so that the following are true: $\mid z_{h_n} \mid\leq (2L\epsilon^{-1} - 1)r \cdot 1_L$, so that along some subsequence (still indexed by n) $z_{h_n} \to z$, $z^i_{h_n} + e^i_{h_n} \geq \epsilon$ and $\frac{p^j_{h_n}}{p^i_{h_n}} \to 0$.

The compactness of T implies that, by passing to a subsequence if necessary we may assume that $(\succeq_{h_n}, e_{h_n}) \to (\succeq, e) \in T$.

Consider for each $n \in N$ the vectors $t_n \in \Re^L_+$ where $t^i_n = -\frac{p^j_{h_n}}{p^i_{h_n}}$, $t^j_n = 1$ and $t^l_n = 0$ for $l \neq i, j$. For these vectors we have that $p_{h_n}t_n = 0$, $|t^i_n| < \epsilon$ for n large enough, $t_n \to t \geq 0$ and $t \neq 0$. By the convexity of preferences it must be that $z_{h_n} + e_{h_n} \succeq_{h_n} z_{h_n} + e_{h_n} + t_n$. Taking limits we conclude that $z + e \succeq z + e + t$ which contradicts the monotonicity of $\succeq$. This contradiction establishes our claim that (9) is true for some $M > 0$. So in this step we can conclude that there exists $M > 0$ so that:

$$\frac{1}{\#H_n}\#\{h \in H_n : \frac{p_h^i}{p_h^j} > M\} < 1 - \frac{\epsilon}{2} \tag{11}$$

Step IV Since $\pi_n^i/\pi_n^j \to \infty$ we have that $\pi_n^i/\pi_n^j(1-\epsilon) > M$ for n large enough. Furthermore, by theorem (1) we have that for n large enough:

$$\frac{1}{\#H_n}\#\left\{h \in H_n : \delta_h(x_n) \leq \epsilon\right\} \geq 1 - \frac{\epsilon}{2}$$

But then for n large enough we have the following string of inequalities

$$\begin{aligned}\frac{1}{\#H_n}\#\left\{h \in H_n : \frac{p_h^i}{p_h^j} > M\right\} \geq & \frac{1}{\#H_n}\#\left\{h \in H_n : \frac{p_h^i}{p_h^j} \geq \frac{\pi_n^i}{\pi_n^j}(1-\epsilon)\right\} \\ \geq & \frac{1}{\#H_n}\#\left\{h \in H_n : \delta_h(x_n) \leq \epsilon\right\} \\ \geq & 1 - \frac{\epsilon}{2}\end{aligned}$$

which contradicts (11). □

We can now prove the following result.

THEOREM 2 *Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in N}$, where $\mathcal{E}_n : H_n \to T$, $\#H_n \to \infty$ and $T \subset \mathcal{P}_{cm} \times [0,s]^L$ is compact. Let $(b_n, q_n) \in \mathbf{N}(\mathcal{E}_n)$ and suppose that for some $\beta > 0$, $\frac{1}{\#H_n}Q_n \gg \beta 1_L$ for n large enough, so that the corresponding $x_n \in \mathcal{N}(\mathcal{E}_n)$ is fully active. Then given any $\epsilon > 0$ there is N so that if $\#H_n > N$ then $\delta_h(x_n) < \epsilon$, $\forall h \in H_n$.*

Proof: Since T is compact by proposition (1), we have that for each $h \in H_n$, $\| z_{n,h} \| = \| x_{n,h} - e_h \| \leq c1_L$ for some $c > 0$.

Furthermore, since for n large enough $\frac{1}{\#H_n}Q_n \gg \beta 1_L$, by passing to a subsequence if necessary we may assume that there is $\xi > 0$ so that $\#\{h \in H_n : q_h^i \geq \xi\} \geq \#H_n\xi$ for all $i = 1, 2, \ldots, L$. In this case, when n is large enough, we have that for all $h \in H_n$:

$$Q_{n,-h}^i = Q_n^i - q_{n,h}^i \geq Q_n^i - s \geq \#H_n\xi^2 - s$$

Fix one $h \in H_n$. We have $\frac{z_{n,h}^i}{Q_{n,-h}^i} \leq \frac{c}{\#H_n\xi^2 - s}$ for all $i = 1, 2, \ldots, L$. It follows that

$$
\begin{aligned}
\mid \frac{p_h^i}{p_h^j} \cdot \frac{\pi^j(\bar{b},\bar{q})}{\pi^i(\bar{b},\bar{q})} - 1 \mid &= \mid \frac{Q_{n,-h}^i(Q_{n,-h}^j - z_{n,h}^j)}{Q_{n,-h}^j(Q_{n,-h}^i - z_{n,h}^i)} - 1 \mid \\
&= \mid \frac{\frac{z_{n,h}^i}{Q_{n,-h}^i}}{1-\frac{z_{n,h}^i}{Q_{n,-h}^i}} - \frac{\frac{z_{n,h}^j}{Q_{n,-h}^j}}{1-\frac{z_{n,h}^i}{Q_{n,-h}^i}} \mid \\
&\leq 2\frac{\frac{c}{\#H_n\xi^2-s}}{1-\frac{c}{\#H_n\xi^2-s}} \\
&= 2\frac{c}{\#H_n\xi^2-s-c}
\end{aligned} \tag{12}
$$

Hence $\delta_h(x_n) = \max\left\{\mid \frac{p_h^i}{p_h^j}\cdot\frac{\pi_n^j}{\pi_n^i} - 1 \mid : i,j = 1,2,\ldots,L\right\} \leq 2\frac{c}{\#H_n\xi^2-s-c}$. Therefore, given $\epsilon > 0$ by choosing $N = \frac{2c+\epsilon(s+c)}{\epsilon\xi^2}$ we have that if $\#H_n > N$ then $\delta_h(x_n) < \epsilon$ for every $h \in H_n$ as desired. □

4 Purely competitive sequences of economies

The results of the previous section can become more transparent by considering sequences of economies converging to a limit. To this end in this section we will consider 'purely competitive' sequences of economies (see Hildenbrand (1974) p.138) which are defined as follows.

Let $T \subset \mathcal{P}_{cm} \times \Re_+^L$ be compact. Consider a sequence $\{\mathcal{E}_n\}_{n\in N}$, where $\mathcal{E}_n : H_n \to T$ such that:

(i) $\#H_n \to \infty$.

(ii) The sequence of distributions of characteristics (μ_n) converges weakly on T.

(iii) If $\mu = \lim \mu_n$ then $\int e \; d\mu_n \to \int e \; d\mu$.

(iv) $\int e \; d\mu > 0$.

Let $\{\mathcal{E}_n\}_{n\in N}$ be such a sequence and consider a sequence of fully active Nash equilibria $x_n \in \mathcal{N}(\mathcal{E}_n)$. Since T is compact, it follows by proposition (1) that $\{x_n\}_{n\in N}$ is uniformly bounded, so we can extract a subsequence (still indexed by n) which converges in distribution, i.e., by defining for each $B \in \Re^L$ $\lambda_n(B) = \frac{1}{\# H_n} \# \{h \in H_n : x_{n,h} \in B\}$ we have that $\lambda_n \to \lambda$ weakly.

Denote now by τ_n the joint distribution of $(\mathcal{E}_n, x_n) : H_n \to T \times \Re^L$. The sequence $(\tau_n)_{n \in N}$ is tight since the sequences of its marginal distributions are tight, so we may assume, by passing to a subsequence if necessary, that $\tau_n \to \tau$ weakly. Hence, this sequence of economies and associated allocations admits a continuous representation (see Hildenbrand (1974) proposition 2 p. 139): there is an atomless measure space $(H, \mathcal{H}, \nu)$, $(\mathcal{E}, x) : H \to T \times \Re^L$ and measurable functions $a_n : H \to H_n$, so that $(\mathcal{E}_n(a_n), x_{n,a_n}) \to (\mathcal{E}, x)$ *ae* in H and the respective distributions of $(\mathcal{E}_n(a_n), x_{n,a_n})$ and $(\mathcal{E}, x)$ are τ_n and τ respectively.

Using this continuous representation, our indicator can be extended in a natural way on H, by $\hat{\delta}_h(x_{n,a_n}) = \delta_{a_n(h)}(x_n)$. The meaning of theorem (1) can be made more transparent as follows:

Lemma 3 *$\hat{\delta}_h(x_{n,a_n}) \to 0$ in measure.*

Proof :

By definition of a continuous representation of the sequence of economies:

$$\begin{aligned}
\nu\left(\left\{h \in H : \hat{\delta}_h(x_{n,a_n}) > \epsilon\right\}\right) &= \nu\left(\left\{h \in H : \delta_{a_n(h)}(x_{n,a_n}) > \epsilon\right\}\right) \\
&= \nu\left(a_n^{-1}\left(\{h \in H_n : \delta_h(x_{n,a_n}) > \epsilon\}\right)\right) \\
&= \frac{1}{\# H_n} \# \{h \in H_n : \delta_h(x_n) > \epsilon\}
\end{aligned}$$

By Theorem (1) the righthand side converges to zero. □

The following proposition establishes that the allocation x is Walrasian for the economy $\mathcal{E}$, provided that the associated sequence of strategic prices does not converge to the boundary of $\Re^L_+$.

Proposition 2 *Let $x_n \in \mathbf{E}(\mathcal{E}_n)$, for each $n \in N$ be fully active and suppose that the sequence of associated strategic market game prices $\{\pi_n\}_{n \in N}$ are such that no subsequence converges to the boundary of $\Re^L_+$. Then $\delta_h(x) = 0$, ae in H.*

Proof :

Normalizing prices so that $\sum_{i=1}^{L} \pi_n^i = 1$, for each $n \in N$, we may assume, by passing to a subsequence if necessary, that $\pi_n \to p > 0$. For $i = 1, 2, \ldots, L$ let $b_h^i = p^i x_h^i$ and $q_h^i = e_h^i$ *ae* in H. It can be verified that $\pi(b, q) = p$ and $e_h + z_h(b, q) = x_h$. Thus, $\delta_h(x)$ is well defined and since $(\mathcal{E}_n(a_n), x_{n,a_n}) \to (\mathcal{E}, x)$ *ae* in H, it follows by continuity of $\hat{\delta}_h(\cdot)$ that $\delta_h(x) = \lim \hat{\delta}_h(x_{n,a_n})$, *ae* in H. By lemma 3 above, $\hat{\delta}_h(x_{n,a_n}) \to 0$ in measure so there is a subsequence $\hat{\delta}_h\left(x_{n_k,a_{n_k}}\right) \to 0$ *ae* in H. Since $\delta_h(x) = \lim \hat{\delta}_h\left(x_{n_k,a_{n_k}}\right)$, *ae* in H, it follows that it must be $\delta_h(x) = 0$, *ae* in H. □

5 Concluding remarks

Notice that the proof of theorem (2) provides a rate of convergence which depends on the set of characteristics T (the constants c and s), but it also depends on the sequence itself (the constant ξ, which in turn depends on β -the uniform lower bound on offers). This is sensible because in strategic market games there is no parameter β that works for all possible sequences: it is possible that some sequences of prices converge to the boundary of the simplex, irrespectively of the set of characteristics. In that case the corresponding sequence of active equilibria converges to one where some markets are inactive, which typically will not be Walrasian. For the same reason a similar qualification on the sequence of Nash equilibria was needed in proposition (2). Hence, the results in this paper must be understood as asserting that the limit of all sequences of Nash equilibria which remain active in the limit is Walrasian.

References

[1] Amir, R., S. Sahi, M. Shubik and S. Yao (1990), A Strategic Market Game with Complete Markets, *Journal of Economic Theory*, **51**, 126-143.

[2] Anderson, R. (1992) The Core in Perfectly Competitive Economies, in Handbook of Game Theory, Vol. I, ed. R.J. Aumann & S. Hart, Elsevier Science Publishers B.V 1992.

[3] Dubey, P. and M. Shubik (1978) A Theory of Money and Financial Institutions. The Non-cooperative Equilibria of a Closed Economy with Market Supply and Bidding Strategies, *Journal of Economic Theory*, **17**, 1-20.

[4] Hildenbrand, W. (1974) Core and Equilibria of a Large Economy, Princeton University Press, Princeton, New Jersey.

[5] Koutsougeras, L.C (2007) Convergence of Strategic Behavior to Price Taking, *Games and Economic Behavior*, **65**, 234-241 (2009).

[6] Mas-Colell, A. (1982) The Cournotian Foundations of Walrasian Equilibrium Theory: An Exposition of Recent Theory, in Advances in Economic Theory, ed. W. Hildenbrand, Cambridge University Press, 183-224.

[7] Peck, J., K. Shell (1989), On the Nonequivalence of the Arrow-Securities Game and the Contingent-Commodities Game, in Barnett,W., Geweke, J., Shell, K., (eds), *Economic complexity: Chaos, sunspots, bubbles, and nonlinearity,* Cambridge University Press, 61-85.

[8] Peck, J., K. Shell (1990), Liquid Markets and Competition, *Games and Economic Behavior*, **2**, 362-377.

[9] Postlewaite, A. and D. Schmeidler (1981) Approximate Walrasian Equilibria and Nearby Economies, *International Economic Review*, **22**, 105-111.

[10] Sahi, S. and S. Yao (1989) The Non-cooperative Equilibria of a Trading Economy with Complete Markets and Consistent Prices, *Journal of Mathematical Economics*, **18**, 325-346.