A switching microstructure model for stock prices

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Abstract

This article proposes a microstructure model for stock prices in which parameters are modulated by a Markov chain determining the market behaviour. In this approach, called the switching microstructure model (SMM), the stock price is the result of the balance between the supply and the demand for shares. The arrivals of bid and ask orders are represented by two mutually- and self-excited processes. The intensities of these processes converge to a mean reversion level that depends upon the regime of the Markov chain. The first part of this work studies the mathematical properties of the SMM. The second part focuses on the econometric estimation of parameters. For this purpose, we combine a particle filter with a Markov Chain Monte Carlo (MCMC) algorithm. Finally, we calibrate the SMM with two and three regimes to daily returns of the S&P500 and compare them with a non switching model.

Keywords: Hawkes process, Switching process, microstructure

1 Introduction

As emphasized in the review of Bouchaud (2010), the market microstructure literature aims to explain the role of market orders on stock prices. The understanding of this relationship has significantly progressed during the last decade. For example, Bouchaud et al. (2009) explain that, because market liquidity may be low, large orders to buy or sell are only traded incrementally, over periods of time as long as weeks. As a result, order flow is a persistent long-memory process. Bacry and Muzy (2014) mention that this persistence of information causes endogeneity in stocks markets and contradicts the classical theory in which prices are driven by an exogenous flow of information.

To duplicate the endogeneity in prices, Bouchaud et al. (2010) propose a model of price fluctuations by generalizing Kyle's approach (1985) according to which the price is the result of the balance (up to a noise term) between bid and ask orders. Cont et al. (2013) study the price impact of order book events using the NYSE TAQ data for 50 U.S. stocks. They show that, over short time

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intervals, price changes are mainly driven by the imbalance between supply and demand. Kelly and Yudovina (2017) model the limit order book on short time scales, where the dynamics are driven by stochastic fluctuations between supply and demand. Horst and Paulsen (2017) study the limit properties of order books.

Bacry et al. (2013 a) reproduce the microstructure noise with multivariate Hawkes processes associated with positive and negative jumps of the asset prices. Bacry et al. (2013 b) characterise the exact macroscopic diffusion limit of this model and show in particular its ability to reproduce empirical stylised fact such as the Epps effect and the lead-lag effect. Bacry and Muzy (2014) extend this approach to prices and volumes with self and mutually excited processes. This particular category of point processes was developed by Hawkes (1971a, b) and Hawkes and Oakes (1974). In its simplest version, the intensity of jumps is persistent and suddenly increases as soon as a jump occurs in the asset price. Bowsher (2002), Hautsch (2004) and Large (2005) illustrate that Hawkes processes capture the dynamics in financial point processes remarkably well. This indicates that the cluster structure implied by the self-exciting nature of these processes provides a reasonable description of the timing structure of events in financial markets. Hardiman and Bouchaud (2014) propose a method to evaluate the integral of the Hawkes kernel, called the branching ratio which is a measure of markets endogeneity. Da Fonseca and Zaatour (2014) provide explicit formulas for moments and the autocorrelation function of the number of jumps over a given interval for the Hawkes process. Based on these moments, they propose an estimation method. Filimonov and Sornette (2015) study the pitfalls in the calibration of Hawkes processes to high frequency data. Jaisson and Rosenbaum (2015) show that nearly unstable Hawkes processes asymptotically behave like integrated Cox–Ingersoll–Ross models. Bacry and Muzy (2016) demonstrate that the jumps correlation matrix of a multivariate Hawkes process is related to the Hawkes kernel matrix by a system of Wiener-Hopf integral equations. This relation is next used to calibrate microstructure models to EuroStoxx (FSXE) and EuroBund (FGBL) future contracts. Bormetti et al. (2015) propose a Hawkes factor model to capture the time clustering of jumps and the high synchronization of jumps across assets. Hainaut (2016 a) introduces clustering of shocks in the dynamic of short term rates with Hawkes processes. Ait-Sahalia et al. (2015) use Hawkes processes to study the level of contagion between stocks markets. Chavez-Demoulin and McGill (2012) model excesses of high-frequency financial time series via a Hawkes process. Hainaut (2016 b) adapts the microstructure model of Bacry and Muzy (2014) to explain the behaviour of swap rates. Lee and Seo (2017) examines the theoretical and empirical perspectives for the symmetric Hawkes model of the price tick structure. Whereas Hainaut (2017) reveals that time changed Lévy processes with self-excited clocks explain the clustering of jumps of S&P 500 and Eurostoxx 50 index. A detailed survey of other applications of Hawkes processes in finance is available in Bacry et al. (2015).

Given that economic cycles influence the trading behaviour and drive up or down the stocks market, we propose a microstructure model allowing for changes of trading dynamics. Our model is an extension of Bacry et al. (2013) in two directions. First, our model allows for regime shifts in the mean reversion level of orders arrivals. Second, the sizes of orders are random positive variables. In this new approach, called the "Switching Microstructure Model" (SMM), each state of the Markov chain represents a particular trading trend. Our approach is also related to the work of Wang et al. (2012) who study an univariate Markov-modulated Hawkes process, with jumps at discrete occurence times. After a complete study of SMM mathematical properties, we develop an estimation procedure that combines a particle filter with a Markov Chain Monte Carlo algorithm. We can draw a parallel between our approach and the research done at a macro-level that emphasizes the strong link between economic cycles and the dynamic of markets. For example, Guidolin and

Timmermann (2005) present evidence of persistent 'bull' and 'bear' regimes in UK stock and bond returns. Guidolin and Timmermann (2008) obtain similar results for international stock markets. Hainaut and MacGilchrist (2012) use Markov-modulated copulas to filter economic cycles in the French stocks and bonds markets. They consider the economic implication of this relation from the perspective of an investor's portfolio allocation. Whereas Al-Anaswah and Wilfing (2011) estimate a two regimes Markov-switching specification of speculative bubbles. Recently, Branger et al. (2014) compares the correlations between asset returns induced by regime switching models with jumps and models with contagious jumps.

The paper proceeds as follows. Section 2 presents the high frequency dynamic of prices. Section 3 proposes closed form and semi-closed form expressions for moments and moment generating functions of jumps intensities and stock prices. The rest of the article focuses on the estimation of SMM parameters. Given that stock prices do not have an analytical probability distribution and that state variables are not observable, the estimation of parameters is done with a particle Markov Chain Monte Carlo algorithm. The SMM with two and three regimes is next fitted on daily data of the S&P 500 index. Our analysis confirms that the switching microstructure market model outperforms its non-switching equivalent version. Furthermore, each regime is clearly identified to a trading trend and to a level of market stress.

2 The switching microstructure model (SMM)

2.1 Stock price

The proposed approach for the analysis of stock prices determination looks at supply and demand in the market. It finds its foundations in the economics theory. In economics, the relationship between the quantity supplied and the price is described by a curve. Under the assumption that all other economic variables are constant, quantities of supplied stocks, noted B_1, increase linearly with prices P_1. The supply curve is then described by the following relation

$$P_1 = L^1 + \zeta_1 B_1 \,,$$

where L^1 and $\zeta_1 > 0$ are respectively the intercept and the elasticity of the supply curve. Under the same assumption, we can derive a demand curve that shows the relationship between the demanded quantities and prices. This demand curve has the usual downward slope, indicating that as the price increases (everything else being equal), the demanded quantity of stocks falls. The equation defining this line is the following

$$P_2 = L^2 - \zeta_2 B_2 \,,$$

where L^2 and ζ_2 are respectively the intercept and the elasticity of the demand curve. The market equilibrium occurs when the demand equals the supply, $B^* = B_1 = B_2$, at a price S such that,

$$S = L^2 - \zeta_2 B^* \quad = \quad L^1 + \zeta_1 B^* . \tag{1}$$

In economics, a change in market conditions is represented by a parallel shift of the demand or supply curve. Mathematically, this shift corresponds to a modification of the intercept L^2 or L^1. In order to model the dynamics of stock prices, L^1 and L^2 are then indexed by the time t and are assumed to be stochastic processes. According to the relation (1), the volume of exchanged stocks at any given time is hence equal to:

$$B_t^* \quad = \quad \frac{L_t^2 - L_t^1}{\zeta_1 + \zeta_2}, \tag{2}$$

and the equilibrium stocks price at time t is given by

$$S_t = \frac{\zeta_1}{\zeta_1 + \zeta_2} L_t^2 - \frac{\zeta_2}{\zeta_1 + \zeta_2} L_t^1. \tag{3}$$

Starting from this theoretical result, we postulate that the stock price S_t is a difference between demand (L_t^2) and supply (L_t^1)

$$S_t = \alpha_2 L_t^2 - \alpha_1 L_t^1 \tag{4}$$

These processes are defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with a right-continuous and complete information filtration $\mathbb{F} = (\mathcal{F}_t)_{t>0}$. P denotes from now on the probability measure. In order to define a realistic microstructure price model while accounting for the impact of market orders as suggested by equation (4), the framework of multivariate switching Hawkes processes is well suited. Inspired from the work of Bacry et al. (2014), the supply and demand quantities that rules S_t are related to numbers and sizes of bid-ask orders.

Let us respectively denote by $T_1^1 < T_2^1 < ...$ and $T_1^2 < T_2^2 < ...$, the sequences of arrival times of supply (bid) and demand (ask) orders. The bid order at time T_n^1 and the ask order at time T_n^2 are defined by random variables $O_n^1 \in \mathcal{F}_{T_n^1}$ and $O_n^2 \in \mathcal{F}_{T_n^2}$. The sequences (T_n^1, O_n^1) and (T_n^2, O_n^2) generate non explosive counting processes $N_t^1 = \sum_{n\geq 1} 1_{\{T_n^1 \leq t\}}$ and $N_t^2 = \sum_{n\geq 1} 1_{\{T_n^2 \leq t\}}$. From now on, L_t^1 and L_t^2 point out the processes modeling the aggregate supply and demand instead of intercepts of demand-supply curves. They are defined as the total of all bid and ask orders till time t which are defined as follows:

$$L_t^1 = \sum_{i=1}^{N_t^1} O_i^1, \tag{5}$$

$$L_t^2 = \sum_{i=1}^{N_t^2} O_i^2. \tag{6}$$

An increase of the aggregate offer of stocks causes a decline of their prices. In the opposite scenario, under the pressure of a high aggregate demand, stocks prices grow up. Then if α_1 and α_2 respectively denotes the permanent impact of bid and ask orders, the economics theory suggests therefore the dynamic (4) for S_t. The order sizes, O_i^1 and O_i^2, are assumed to be identically independent (i.i.d.) positive random variables: $O_i^1 \sim O^1$ and $O_i^2 \sim O^2$. The assumption of independence between sizes cannot be checked statistically as we do not have information about volumes. However this assumption is common in the literature about microstructure, as e.g. in Bacry et al. (2014). The densities of supply and demand orders are denoted by $\nu_1(z)$ and $\nu_2(z)$ and defined on $(0, \infty)$ sich that the moment generating function of orders, noted $\psi_i(\omega) := \mathbb{E}(e^{\omega O_i})$ exists and is finite for $\omega \in \mathbb{C}$. First and second moments exists and are denoted $\mu_1 = \mathbb{E}(O^1)$, $\mu_2 = \mathbb{E}(O^2)$, $\eta_1 = \mathbb{E}\left(\left(O^1\right)^2\right)$, $\eta_2 = \mathbb{E}\left(\left(O^2\right)^2\right)$.

2.2 Economic regimes

At this stage, stock prices in this model are not explicitly mean reverting. Then, there is no warranty that prices do not diverge at long term to extreme positive or negative values. However, we will see that such a divergence can be avoided by introducing dependence between arrivals of bid and ask orders. If new bid (resp. ask) orders raise the probability of ask (resp. bid) order arrivals, we expect a stable behaviour for $(S_t)_{t\geq 0}$. Mathematically, the mutual- and self-excitation is obtained by assuming that intensities are driven by a bivariate Hawkes process but we will come back on this point later. On the other hand, flows of information and economic cycles influence the demand and supply for stocks at macro level. To introduce such a feature in our model, we assume that the economic information is carried by a hidden Markov chain with a finite number of regimes, noted l. The chain is a vector process $(\theta_t)_{t\geq 0}$ taking values from a set of $\mathbb{R}^l$-valued unit vectors $E = \{e_1, \ldots, e_l\}$, where $e_j = (0, \ldots, 1, \ldots, 0)'$. The filtration generated by $(\theta_t)_{t\geq 0}$ is denoted by $(\mathcal{G}_t)_{t\geq 0}$ and is a subfiltration of $(\mathcal{F}_t)_t$. The set of regimes is denoted by $\mathcal{N} := \{1, 2, \cdots, l\}$. The generator of θ_t is an $l \times l$ matrix $Q_0 := (q_{i,j})_{i,j=1,2,\ldots,l}$ containing the instantaneous probabilities of transition. They satisfy the following standard conditions:

$$q_{i,j} \geq 0, \quad \forall i \neq j, \quad \text{and} \quad \sum_{j=1}^{l} q_{i,j} = 0, \quad \forall i \in \mathcal{N}. \tag{7}$$

If Δ is a small interval of time, $q_{i,j}\Delta$ is close to the probability that the Markov chain transits from state i to state j, with $i \neq j$. Whereas $1 - q_{i,i}\Delta$ approaches the probability that the chain stays in state i. The matrix of transition probabilities over the time interval $[t, s]$ is denoted as $P(t, s)$ and is the matrix exponential of the generator matrix, times the length of the time interval:

$$P(t, s) = \exp\left(Q_0(s - t)\right), \quad s \geq t. \tag{8}$$

The elements of this matrix , $p_{i,j}(t, s)$, $i, j \in \mathcal{N}$, defined as

$$p_{i,j}(t, s) = P(\theta_s = e_j \,|\, \theta_t = e_i), \quad i, j \in \mathcal{N}, \tag{9}$$

are the probabilities of switching from state i at time t to state j at time s. The probability of the chain being in state i at time t, denoted by $p_i(t)$, depends upon the initial probabilities $p_k(0)$ at time $t = 0$ and the transition probabilities $p_{k,i}(0, t)$, where $k = 1, 2, \ldots, l$, as follows:

$$p_i(t) = P(\theta_t = e_i) = \sum_{k=1}^{l} p_k(0) p_{k,i}(0, t), \quad \forall i \in \mathcal{N}. \tag{10}$$

The stationary distribution of the Markov chain is denoted Π and is defined by the next limit

$$\Pi \quad = \quad \lim_{t\to\infty} \exp\left(Q_0 t\right) \,.$$

These stationary probabilities will play an important role in the calculation of the stock equilibrium price.

2.3 Order arrival intensities

We propose to specify the processes N_t^1 , N_t^2, directly through their conditional arrival rates or intensities, $\left(\lambda_t^1\right)_{t\geq 0}$ and $\left(\lambda_t^2\right)_{t\geq 0}$. We assume that intensities λ_t^1 and λ_t^2 of order arrivals (OAI) are processes defined on a subfiltration $\mathcal{H}_t \subset \mathcal{F}_t$ governed by the next equations:

$$d\lambda_t^i \quad = \quad \kappa_i(c_{i,t} - \lambda_t^i)dt + \delta_{i,1} dL_t^1 + \delta_{i,2} dL_t^2 \quad i = 1, 2, \tag{11}$$

where $\delta_{i,j}$ for $i, j = 1, 2$, are constant. Coefficients $\delta_{1,2} \in \mathbb{R}^+$ and $\delta_{2,1} \in \mathbb{R}^+$ set the cross impact of demand on supply and vice versa. They measure the dependence between them and can capture some interesting stylized facts. E.g. if $\delta_{12} > 0$, the frequency of bid orders increases when the demand, L_t^2, steps up and drives up stock prices according to equation (4). Coefficients $\delta_{1,1}$ and $\delta_{2,2}$ set the self-excitation levels. The levels of mean reversion of OAI, $c_{i,t}$ for $i = 1, 2$, are modulated by the Markov chain θ_t representative of the economic regime and introduced in the previous section:

$$c_{i,t} = c_i^\top \theta_t \,.$$

where $c_{i=1,2}$ are two strictly positive $l-$vectors: $c_i = (c_{i,1}, ..., c_{i,l})^\top$. $\kappa_i \in \mathbb{R}^+$ for $i = 1, 2$ are the speeds of mean reversion of intensities. $(\kappa_i)_{i=1,2}$ and coefficients $(\delta_{i,j})_{i,j=1,2}$ are not modulated by the Markov chain for several reasons. This hypothesis aims to preserve the parsimony and the analytical tractability of our model. From an economic point of view, this assumption implies that the market always adjusts to new conditions with the same velocity. From a technical point of view, we will see later that modulating other parameters than c makes processes non Markov.

To summarize, we use three filtrations in the following developments. The first filtration $(\mathcal{G}_t)_{t\geq 0}$ is generated by $(\theta_t)_{t\geq 0}$. The second filtration $(\mathcal{H}_t)_{t\geq 0}$ is the collection of sigma-algebras carrying exclusively the information about intensity processes: $\mathcal{H}_t = \sigma\left(\lambda_u^1, \lambda_u^2 \,:\, u \leq t\right)$. Observing $\mathcal{H}_t$ informs us about the values of intensities up to time t. Conditionally to $\mathcal{H}_t$, the jump processes $\left(N_t^1\right)_{t\geq 0}$ and $\left(N_t^2\right)_{t\geq 0}$ are non-homogeneous Poisson processes with $\mathcal{H}_{t-}$-adapted intensities λ_t^1 and λ_t^2. The filtration $\mathcal{H}_t$ is not independent from $\mathcal{G}_t$ as intensities depends upon the evolution of θ_t. Notice however that $\mathcal{G}_t$ contains less information than $\mathcal{H}_t$ about the intensities. Therefore $\left(N_t^1\right)_{t\geq 0}$ and $\left(N_t^2\right)_{t\geq 0}$ remains still non-homogeneous Poisson processes conditionally to $\mathcal{H}_t \vee \mathcal{G}_t$.

The third filtration $(\mathcal{F}_t)_{t\geq 0}$ is the global filtration. This is the collection of sigma-algebras generated by all processes:

$$\mathcal{F}_t = \sigma\left(\lambda_u^1, L_u^1, N_u^1, \lambda_u^2, L_u^2, N_u^2, \theta_u, S_u \,:\, u \leq t\right) \,.$$

By definition, the filtrations $(\mathcal{G}_t)_{t\geq 0}$ and $(\mathcal{H}_t)_{t\geq 0}$ are included in $\mathcal{F}_t$. From equation (11), we infer the following lemma:

Lemma 2.1. *Under the assumption that λ_t^1 and λ_t^2 are driven by the SDE (11) and that $\lambda_0^1 > 0$, $\lambda_0^2 > 0$, order arrival intensities (OAI) are strictly positive processes equal to:*

$$\begin{aligned} \lambda_t^i &= \lambda_0^i - \kappa_i \int_0^t e^{\kappa_i (s-t)} \left(\lambda_0^i - c_{i,s}\right) ds \\ &\quad + \int_0^t \delta_{i,1} e^{\kappa_i (s-t)} dL_s^1 + \int_0^t \delta_{i,2} e^{\kappa_i (s-t)} dL_s^2 \qquad i = 1, 2. \end{aligned} \tag{12}$$

The proof is reported in appendix. From equation (12) we can show that λ_t^i is related to λ_s^i for any $s \leq t$ as follows:

$$\begin{aligned} \lambda_t^i &= \lambda_s^i - \kappa_i \int_s^t e^{\kappa_i (u-t)} \left(\lambda_s^i - c_{i,u}\right) du \\ &\quad + \int_s^t \delta_{i,1} e^{\kappa_i (u-t)} dL_u^1 + \int_s^t \delta_{i,2} e^{\kappa_i (u-t)} dL_u^2 \qquad i = 1, 2. \end{aligned} \tag{13}$$

The next section explores the properties of intensities, orders counting and cumulated orders processes. We will first show that their moments exist and shows that $\left(\lambda_t^1, L_t^1, N_t^1, \lambda_t^2, L_t^2, N_t^2, \theta_t\right)_{t\geq 0}$ is a $\mathcal{F}_t$-Markov process.

3 Main properties

This section explores the mathematical features of the Switching Microstructure Model (SMM). The first subsection presents the first and second moments of orders arrival intensities (OAI). The second subsection studies the expected stock price and its asymptotic limit. Whereas the last subsection focuses on the probability generating and moment generating functions.

3.1 Moments of Order Arrival Intensities (OAI)

The expected intensities, conditionally to the sample path of the hidden Markov chain (information carried by the augmented filtration $\mathcal{F}_s \vee \mathcal{G}_t$ with $s \leq t$), are provided in the following proposition. This result is next used to deduce their expectations with respect to the smaller filtration $\mathcal{F}_s$.

Proposition 3.1. *Let us denote by γ_1 and γ_2 the following real numbers:*

$$\begin{aligned}\gamma_1 &:= \frac{1}{2}\left((\delta_{1,1}\mu_1 - \kappa_1) + (\delta_{2,2}\mu_2 - \kappa_2)\right) + \\ &\quad \frac{1}{2}\sqrt{\left((\delta_{1,1}\mu_1 - \kappa_1) - (\delta_{2,2}\mu_2 - \kappa_2)\right)^2 + 4\delta_{1,2}\delta_{2,1}\mu_1\mu_2}, \\ \gamma_2 &= \frac{1}{2}\left((\delta_{1,1}\mu_1 - \kappa_1) + (\delta_{2,2}\mu_2 - \kappa_2)\right) - \\ &\quad \frac{1}{2}\sqrt{\left((\delta_{1,1}\mu_1 - \kappa_1) - (\delta_{2,2}\mu_2 - \kappa_2)\right)^2 + 4\delta_{1,2}\delta_{2,1}\mu_1\mu_2}\,.\end{aligned} \tag{14}$$

Conditionally to $\mathcal{F}_s \vee \mathcal{G}_t$ with $s \leq t$, the processes λ_t^i are Markov and their expected value of λ_t^i is given by the next expression:

$$\begin{aligned}\begin{pmatrix} \mathbb{E}\left(\lambda_t^1 \,|\, \mathcal{F}_s \vee \mathcal{G}_t\right) \\ \mathbb{E}\left(\lambda_t^2 \,|\, \mathcal{F}_s \vee \mathcal{G}_t\right) \end{pmatrix} &= V \int_s^t \begin{pmatrix} e^{\gamma_1(t-u)} & 0 \\ 0 & e^{\gamma_2(t-u)} \end{pmatrix} V^{-1} \begin{pmatrix} \kappa_1 c_{1,u} \\ \kappa_2 c_{2,u} \end{pmatrix} du \\ &\quad + V \begin{pmatrix} e^{\gamma_1(t-s)} & 0 \\ 0 & e^{\gamma_2(t-s)} \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_s^1 \\ \lambda_s^2 \end{pmatrix},\end{aligned} \tag{15}$$

where V ,V^{-1} are matrices given by:

$$V = \begin{pmatrix} -\delta_{1,2}\mu_2 & -\delta_{1,2}\mu_2 \\ (\delta_{1,1}\mu_1 - \kappa_1) - \gamma_1 & (\delta_{1,1}\mu_1 - \kappa_1) - \gamma_2 \end{pmatrix}, \tag{16}$$

$$V^{-1} = \frac{1}{\Upsilon} \begin{pmatrix} (\delta_{1,1}\mu_1 - \kappa_1) - \gamma_2 & \delta_{1,2}\mu_2 \\ \gamma_1 - (\delta_{1,1}\mu_1 - \kappa_1) & -\delta_{1,2}\mu_2 \end{pmatrix}, \tag{17}$$

and $\Upsilon \in \mathbb{R}$ is the determinant of V defined by

$$\Upsilon := -\delta_{1,2}\mu_2\sqrt{\left((\delta_{1,1}\mu_1 - \kappa_1) - (\delta_{2,2}\mu_2 - \kappa_2)\right)^2 + 4\delta_{1,2}\delta_{2,1}\mu_1\mu_2}. \tag{18}$$

The proof is reported in appendix. Knowing the expectation of λ_t^i conditionally to the sample path of θ_t, we can infer the unconditional expectations of intensities and prove that λ_t^i are Markov as stated in the following proposition.

Proposition 3.2. *The expected values of* λ_t^1 *and* λ_t^2 *conditionally to* $\mathcal{F}_s$ *for* $s \leq t$, *are given by the next expression:*

$$\begin{pmatrix} \mathbb{E}\left(\lambda_t^1 \,|\, \mathcal{F}_s\right) \\ \mathbb{E}\left(\lambda_t^2 \,|\, \mathcal{F}_s\right) \end{pmatrix} = V \begin{pmatrix} m_1(t,\theta_s) \\ m_2(t,\theta_s) \end{pmatrix} + V \begin{pmatrix} e^{\gamma_1(t-s)} & 0 \\ 0 & e^{\gamma_2(t-s)} \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_s^1 \\ \lambda_s^2 \end{pmatrix}, \tag{19}$$

where $m_1(t,\theta_s)$ *and* $m_2(t,\theta_s)$ *are respectively equal to*

$$m_1(t,\theta_s) = e_1^\top M_1(t,\theta_s) \begin{pmatrix} \frac{1}{\Upsilon}\kappa_1\left((\delta_{1,1}\mu_1 - \kappa_1) - \gamma_2\right) \\ \frac{1}{\Upsilon}\kappa_2\delta_{1,2}\mu_2 \end{pmatrix}, \tag{20}$$

$$m_2(t,\theta_s) = e_2^\top M_1(t,\theta_s) \begin{pmatrix} \frac{1}{\Upsilon}\kappa_1\left(\gamma_1 - (\delta_{1,1}\mu_1 - \kappa_1)\right) \\ -\frac{1}{\Upsilon}\kappa_2\delta_{1,2}\mu_2 \end{pmatrix}, \tag{21}$$

and $M_1(t,\theta_s)$ *is the following time and state dependent matrix:*

$$M_1(t,\theta_s) := \begin{pmatrix} \theta_s^\top (Q_0 - \gamma_1 I)^{-1} & 0 \\ 0 & \theta_s^\top (Q_0 - \gamma_2 I)^{-1} \end{pmatrix} \times \begin{pmatrix} \left[\exp\left(Q_0(t-s)\right) - \exp\left(I\gamma_1(t-s)\right)\right] & 0 \\ 0 & \left[\exp\left(Q_0(t-s)\right) - \exp\left(I\gamma_2(t-s)\right)\right] \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1 & c_2 \end{pmatrix}. \tag{22}$$

Here, I *points out the identity matrix of size* $l \times l$.

See the appendix for the proof. Notice that we cannot find moments and prove the Markov feature of $\left(\lambda_t^1, L_t^1, N_t^1, \lambda_t^2, L_t^2, N_t^2, \theta_t\right)_{t\geq 0}$ when the speed reversion or mutual excitation parameters are modulated by θ_t. To understand this point, let us assume that κ_1 and κ_2 depends on θ_t . In this case, the matrix V and parameters γ_1, γ_2 involved in conditional expectations of intensities with respect to $\mathcal{F}_s \vee \mathcal{G}_s$ (proposition 3.1) are modulated by θ_t. Calculating the expectation with respect to the filtration $\mathcal{F}_s$, as done in the proof of proposition 3.2 is in this case no more possible. Mainly because this requires to calculate the expectation of an integral of a product of terms related to V, V^{-1}, γ_1, γ_2 , $c_{1,t}$ and $c_{2,t}$ that all are modulated by θ_t. If we want to preserve the Markov feature of our model, the drift is the only modulable parameter. Wang et al. (2012) draw the same conclusion for an univariate switching Hawkes process.

From this last proposition, we infer the conditions that ensure the stability of the process. The OAI's remain finite ($\lambda_t^1 < \infty$ and $\lambda_t^2 < \infty$ almost surely $\forall t \geq 0$) if only γ_1 and γ_2 are negative. In the opposite case, the limits of λ_t^i for $i = 1, 2$ when $t \to \infty$ diverge to $+\infty$. If $\gamma_1 < 0$ and $\gamma_2 < 0$, the expected intensities converge toward:

$$\lim_{t\to\infty} \begin{pmatrix} \mathbb{E}\left(\lambda_t^1 \,|\, \mathcal{F}_s\right) \\ \mathbb{E}\left(\lambda_t^2 \,|\, \mathcal{F}_s\right) \end{pmatrix} = V \begin{pmatrix} m_1(\infty,\theta_s) \\ m_2(\infty,\theta_s) \end{pmatrix}, \tag{23}$$

where the constant $m_1(\infty)$ and $m_2(\infty)$ are the limits of functions $m_2(t,\theta_s)$ and $m_2(t,\theta_s)$:

$$\begin{aligned} m_1(\infty,\theta_s) := \lim_{t\to\infty} m_1(t,\theta_s) &= \frac{1}{\Upsilon}\Big[\kappa_1\left((\delta_{1,1}\mu_1 - \kappa_1) - \gamma_2\right)\theta_s^\top (Q_0 - \gamma_1 I)^{-1}\,\Pi\, c_1 \\ &\quad + \kappa_2\delta_{1,2}\mu_2\theta_s^\top (Q_0 - \gamma_1 I)^{-1}\,\Pi\, c_2\Big], \\ m_2(\infty,\theta_s) := \lim_{t\to\infty} m_2(t,\theta_s) &= \frac{1}{\Upsilon}\Big[\kappa_1\left(\gamma_1 - (\delta_{1,1}\mu_1 - \kappa_1)\right)\kappa_1\theta_s^\top (Q_0 - \gamma_2 I)^{-1}\,\Pi\, c_1 \\ &\quad - \kappa_2\delta_{1,2}\mu_2\theta_s^\top (Q_0 - \gamma_2 I)^{-1}\,\Pi\, c_2\Big], \end{aligned}$$

and $\Pi = \lim_{t\to\infty} \exp(Q_0 t)$ is the stationary distribution of θ_t. From propositions 15 and 3.2, we infer the following result:

Corollary 3.3. $\left(\lambda_t^1, L_t^1, N_t^1, \lambda_t^2, L_t^2, N_t^2, \theta_t\right)_{t\geq 0}$ *is a $\mathcal{F}_t$-Markov process in the state space*

$$D = \left(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{N}\right)^2 \times E\,.$$

The expectations of L_t^1 and L_t^2 admit closed form expressions that are developed in Section 3.2. Let us denote by $J_t^i = (L_t^i, N_t^i)$ for $i = 1, 2$ the bivariate processes related to arrivals of orders. The multivariate process $(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, \theta_t)$ is a Markov process adapted to $\mathcal{F}$ with càdlag paths. By construction, it is decomposable and then a semi-martingale. The Itô's formula for semi-martingales (see e.g. Protter 2004, theorem 32, p79), allows us to find the infinitesimal generator for any function $g : D \to \mathbb{R}$ with continuous partial derivatives g_{λ_1} , g_{λ_2}. If $\theta_t = e_i$, the generator of this function, denoted $\mathcal{A}g(.)$, is given by the following expression:

$$\begin{aligned}\mathcal{A}g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_i) &= \kappa_1(c_1^\top e_i - \lambda_t^1)g_{\lambda^1} + \kappa_2(c_2^\top e_i - \lambda_t^2)g_{\lambda^2} \\ &+\lambda_t^1 \int_{-\infty}^{+\infty} g(\lambda_t^1 + \delta_{1,1}z, J_t^1 + (z,1)^\top, \lambda_t^2 + \delta_{2,1}z, J_t^2, e_i) - g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_i)\nu_1(dz) \\ &+\lambda_t^2 \int_{-\infty}^{+\infty} g(\lambda_t^1 + \delta_{1,2}z, J_t^1, \lambda_t^2 + \delta_{2,2}z, J_t^2 + (z,1)^\top, e_i) - g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_i)\nu_2(dz) \\ &+\sum_{j\neq i}^{l} q_{i,j}\left(g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_j) - g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_i)\right)\,.\end{aligned} \tag{24}$$

Under mild conditions, the expectation of $g(.)$ is equal to the integral of the expected infinitesimal generator. Using the Fubini's theorem leads to the following result:

$$\begin{aligned}&\mathbb{E}\left(g(\lambda_T^1, J_T^1, \lambda_T^2, J_T^2, \theta_T)|\mathcal{F}_t\right) \\ &= g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2) + \int_t^T \mathbb{E}\left(\mathcal{A}g(\lambda_s^1, J_s^1, \lambda_s^2, J_s^2, \theta_s)|\mathcal{F}_t\right) ds.\end{aligned} \tag{25}$$

The derivative of this expectation with respect to time is equal to its expected infinitesimal generator:

$$\frac{\partial}{\partial T}\mathbb{E}\left(g(\lambda_T^1, J_T^1, \lambda_T^2, J_T^2, , \theta_T)|\mathcal{F}_t\right) = \mathbb{E}\left(\mathcal{A}g(\lambda_T^1, J_T^1, \lambda_T^2, J_T^2, \theta_T)|\mathcal{F}_t\right). \tag{26}$$

We use this result later in the proof of Proposition 3.6. The remainder of this paragraph is devoted to the calculation of the variance of intensities λ_t^1 and λ_t^2. Unfortunately, the variances of these OAI do not admit any closed form expression. But the second order moment of λ_t^i can be calculated numerically by solving a system of ordinary differential equations (ODE). Writing this system requires additional intermediate results about the expectations of mean reversion levels and OAI. The next proposition (proven in appendix) presents these expectations when the sample path of θ_t is observed from 0 up to time t.

Proposition 3.4. *The expected value of $c_{j,t}\lambda_t^i$ for $i,j=1,2$ with respect to the augmented filtration $\mathcal{F}_0 \vee \mathcal{G}_t$ is given by*

$$\begin{pmatrix} \mathbb{E}\left(c_{1,t}\lambda_t^1|\mathcal{F}_0\vee\mathcal{G}_t\right) \\ \mathbb{E}\left(c_{2,t}\lambda_t^1|\mathcal{F}_0\vee\mathcal{G}_t\right) \\ \mathbb{E}\left(c_{1,t}\lambda_t^2|\mathcal{F}_0\vee\mathcal{G}_t\right) \\ \mathbb{E}\left(c_{2,t}\lambda_t^2|\mathcal{F}_0\vee\mathcal{G}_t\right) \end{pmatrix} = \int_0^t W\exp\left(F\,s\right)\,W^{-1}K\begin{pmatrix} c_{1,s}^2 \\ c_{2,s}^2 \\ c_{1,s}c_{2,s} \end{pmatrix} ds + W\exp\left(F\,t\right)W^{-1}\begin{pmatrix} c_{1,0}\lambda_0^1 \\ c_{2,0}\lambda_0^1 \\ c_{1,0}\lambda_0^2 \\ c_{2,0}\lambda_0^2 \end{pmatrix}, \tag{27}$$

where W is a 4×4 matrix

$$W=\begin{pmatrix} -\delta_{1,2}\mu_2 & -\delta_{1,2}\mu_2 & 0 & 0 \\ 0 & 0 & -\delta_{1,2}\mu_2 & -\delta_{1,2}\mu_2 \\ (\delta_{1,1}\mu_1-\kappa_1)-\gamma_1 & (\delta_{1,1}\mu_1-\kappa_1)-\gamma_2 & 0 & 0 \\ 0 & 0 & (\delta_{1,1}\mu_1-\kappa_1)-\gamma_1 & (\delta_{1,1}\mu_1-\kappa_1)-\gamma_2 \end{pmatrix},$$

that admits the following inverse

$$W^{-1}=\frac{1}{\Upsilon}\begin{pmatrix} (\delta_{1,1}\mu_1-\kappa_1)-\gamma_2 & 0 & \delta_{1,2}\mu_2 & 0 \\ \gamma_1-(\delta_{1,1}\mu_1-\kappa_1) & 0 & -\delta_{1,2}\mu_2 & 0 \\ 0 & (\delta_{1,1}\mu_1-\kappa_1)-\gamma_2 & 0 & \delta_{1,2}\mu_2 \\ 0 & \gamma_1-(\delta_{1,1}\mu_1-\kappa_1) & 0 & -\delta_{1,2}\mu_2 \end{pmatrix}.$$

Υ is still defined by equation (18) whereas F and K are the following matrix

$$F=\begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_2 \end{pmatrix} \quad , \quad K=\begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_1 \\ 0 & 0 & \kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix}.$$

γ_1 and γ_2 are defined by equations (14).

Using similar arguments to these used in the proof of proposition 3.2, we infer the unconditional expectations of the product of mean reversion levels and of intensities:

Proposition 3.5. *Let us denote $\bar{c}_1^2=\left(c_{1,1}^2,\ldots,c_{1,l}^2\right)$, $\bar{c}_2^2=\left(c_{2,1}^2,\ldots,c_{2,l}^2\right)$ and*

$$\bar{c}_{1\times 2}=\left(c_{1,1}\times c_{2,1},\ldots,c_{1,l}\times c_{2,l}\right).$$

Expectations of $c_{j,t}\lambda_t^i$ for $i,j=1,\,2$ with respect to the $\mathcal{F}_0$ are equal to

$$\begin{pmatrix} \mathbb{E}\left(c_{1,t}\lambda_t^1|\mathcal{F}_0\right) \\ \mathbb{E}\left(c_{2,t}\lambda_t^1|\mathcal{F}_0\right) \\ \mathbb{E}\left(c_{1,t}\lambda_t^2|\mathcal{F}_0\right) \\ \mathbb{E}\left(c_{2,t}\lambda_t^2|\mathcal{F}_0\right) \end{pmatrix} = W\ \left(X\left(t,\theta_0\right)+Y(t,\theta_0)\right)+W\exp\left(Ft\right)W^{-1}\begin{pmatrix} c_{1,0}\lambda_0^1 \\ c_{2,0}\lambda_0^1 \\ c_{1,0}\lambda_0^2 \\ c_{2,0}\lambda_0^2 \end{pmatrix},$$

where $X(t,\theta_0)$ and $Y(t,\theta_0)$ are the next time-dependent vectors of dimension 4:

$$X(t,\theta_0)=\frac{1}{\Upsilon}\begin{pmatrix}\kappa_1\left((\delta_{1,1}\mu_1-\kappa_1)-\gamma_2\right)\left(\theta_0\left(\frac{\left(e^{(Q_0+\gamma_1 I)t}-I\right)}{Q_0+\gamma_1 I}\right)c_1^2\right)\\ \kappa_1\left(\gamma_1-(\delta_{1,1}\mu_1-\kappa_1)\right)\left(\theta_0\left(\frac{\left(e^{(Q_0+\gamma_2 I)t}-I\right)}{Q_0+\gamma_2 I}\right)c_1^2\right)\\ \kappa_1\left((\delta_{1,1}\mu_1-\kappa_1)-\gamma_2\right)\left(\theta_0\left(\frac{\left(e^{(Q_0+\gamma_1 I)t}-I\right)}{Q_0+\gamma_1 I}\right)\bar{c}_{1,2}\right)\\ \kappa_1\left(\gamma_1-(\delta_{1,1}\mu_1-\kappa_1)\right)\left(\theta_0\left(\frac{\left(e^{(Q_0+\gamma_2 I)t}-I\right)}{Q_0+\gamma_2 I}\right)\bar{c}_{1,2}\right)\end{pmatrix},$$

$$Y(t,\theta_0)=\frac{1}{\Upsilon}\begin{pmatrix}\delta_{1,2}\mu_2\kappa_2\left(\theta_0\left(\frac{\left(e^{(Q_0+\gamma_1 I)t}-I\right)}{Q_0+\gamma_1 I}\right)\bar{c}_{1,2}\right)\\ -\delta_{1,2}\mu_2\kappa_2\left(\theta_0\left(\frac{\left(e^{(Q_0+\gamma_2 I)t}-I\right)}{Q_0+\gamma_2 I}\right)\bar{c}_{1,2}\right)\\ \delta_{1,2}\mu_2\kappa_2\left(\theta_0\left(\frac{\left(e^{(Q_0+\gamma_1 I)t}-I\right)}{Q_0+\gamma_1 I}\right)\bar{c}_2^2\right)\\ -\delta_{1,2}\mu_2\kappa_2\left(\theta_0\left(\frac{\left(e^{(Q_0+\gamma_2 I)t}-I\right)}{Q_0+\gamma_2 I}\right)\bar{c}_2^2\right)\end{pmatrix}.$$

As announced earlier, the last result of this subsection presents the ODE's satisfied by the second order moments of intensities. Solving them numerically allows us to evaluate the standard deviation and correlation of intensities.

Proposition 3.6. *The second order moments of λ_t are solution of a system of ODE:*

$$\begin{pmatrix}\frac{\partial}{\partial t}\mathbb{E}\left(\left(\lambda_t^1\right)^2\mid\mathcal{F}_0\right)\\ \frac{\partial}{\partial t}\mathbb{E}\left(\left(\lambda_t^2\right)^2\mid\mathcal{F}_0\right)\\ \frac{\partial}{\partial t}\mathbb{E}\left(\lambda_t^1\lambda_t^2\mid\mathcal{F}_0\right)\end{pmatrix}=\begin{pmatrix}2\kappa_1&0&0&0\\0&0&0&2\kappa_2\\0&\kappa_2&\kappa_1&0\end{pmatrix}\begin{pmatrix}\mathbb{E}\left(c_{1,t}\lambda_t^1|\mathcal{F}_0\right)\\ \mathbb{E}\left(c_{2,t}\lambda_t^1|\mathcal{F}_0\right)\\ \mathbb{E}\left(c_{1,t}\lambda_t^2|\mathcal{F}_0\right)\\ \mathbb{E}\left(c_{2,t}\lambda_t^2|\mathcal{F}_0\right)\end{pmatrix}+$$

$$\begin{pmatrix}\delta_{1,1}^2\eta_1&\delta_{1,2}^2\eta_2\\ \delta_{2,1}^2\eta_1&\delta_{2,2}^2\eta_2\\ \delta_{1,1}\delta_{2,1}\eta_1&\delta_{1,2}\delta_{2,2}\eta_2\end{pmatrix}\begin{pmatrix}\mathbb{E}\left(\lambda_t^1\mid\mathcal{F}_0\right)\\ \mathbb{E}\left(\lambda_t^2\mid\mathcal{F}_0\right)\end{pmatrix}+$$

$$\begin{pmatrix}2\left(\delta_{1,1}\mu_1-\kappa_1\right)&0&2\delta_{1,2}\mu_2\\0&2\left(\delta_{2,2}\mu_2-\kappa_2\right)&2\delta_{2,1}\mu_1\\ \delta_{2,1}\mu_1&\delta_{1,2}\mu_2&\begin{pmatrix}\delta_{1,1}\mu_1-\kappa_1\\+\delta_{2,2}\mu_2-\kappa_2\end{pmatrix}\end{pmatrix}\begin{pmatrix}\mathbb{E}\left(\left(\lambda_t^1\right)^2\mid\mathcal{F}_0\right)\\ \mathbb{E}\left(\left(\lambda_t^2\right)^2\mid\mathcal{F}_0\right)\\ \mathbb{E}\left(\lambda_t^1\lambda_t^2\mid\mathcal{F}_0\right)\end{pmatrix},$$

with the initial conditions $\mathbb{E}\left(\left(\lambda_0^1\right)^2\mid\mathcal{F}_0\right)=\left(\lambda_0^1\right)^2$, $\mathbb{E}\left(\left(\lambda_0^2\right)^2\mid\mathcal{F}_0\right)=\left(\lambda_0^2\right)^2$ *and* $\mathbb{E}\left(\lambda_t^1\lambda_t^2\mid\mathcal{F}_0\right)=\lambda_0^1\lambda_0^2$.

We refer the reader to the appendix for the proof of this result, which is based on relation (26). The next subsection studies the equilibrium price of stocks, such as defined by equation (4) .

3.2 Stock price

This subsection presents the expectation, the asymptotic limit and the moment generating function of stock prices. Remember that in the SMM, the price is determined by the equilibrium between the aggregate supply and offer as follows:

$$S_t = \alpha_2 \int_0^t dL_s^2 - \alpha_1 \int_0^t dL_s^1 .$$

Given that intensities at time t^- are independent from jumps at time t, then the expected stock price is equal to

$$\mathbb{E}\left(S_t|\mathcal{F}_0\right) = \alpha_2\mu_2 \int_0^t \mathbb{E}\left(\lambda_s^2|\mathcal{F}_0\right) ds - \alpha_1\mu_1 \int_0^t \mathbb{E}\left(\lambda_s^1|\mathcal{F}_0\right) .$$

If we insert the expressions (3.2) of the conditionial expectations, we prove by direct integration that the expected price, as stated in the following proposition.

Proposition 3.7. *The expected stock price in the SMM is equal to:*

$$\mathbb{E}\left(S_t|\mathcal{F}_0\right) = S_0 + \begin{pmatrix} -\alpha_1\mu_1 \\ \alpha_2\mu_2 \end{pmatrix}^\top V \begin{pmatrix} \int_0^t m_1\left(s,\theta_0\right) ds \\ \int_0^t m_2\left(s,\theta_0\right) ds \end{pmatrix} \tag{28}$$

$$+ \begin{pmatrix} -\alpha_1\mu_1 \\ \alpha_2\mu_2 \end{pmatrix}^\top V \begin{pmatrix} \frac{1}{\gamma_1}\left(e^{\gamma_1 t}-1\right) & 0 \\ 0 & \frac{1}{\gamma_2}\left(e^{\gamma_2 t}-1\right) \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_0^1 \\ \lambda_0^2 \end{pmatrix} .$$

The integrals $\int_0^t m_1\left(s,\theta_0\right) ds$ *and* $\int_0^t m_2\left(s,\theta_0\right) ds$ *are respectively given by*

$$\int_0^t m_1\left(s,\theta_0\right) ds = e_1^\top M_2(t,\theta_0) \begin{pmatrix} \frac{1}{\Upsilon}\kappa_1\left(\left(\delta_{1,1}\mu_1-\kappa_1\right)-\gamma_2\right) \\ \frac{1}{\Upsilon}\kappa_2\delta_{1,2}\mu_2 \end{pmatrix} \tag{29}$$

$$\int_0^t m_2\left(s,\theta_0\right) ds = e_2^\top M_2(t,\theta_0) \begin{pmatrix} \frac{1}{\Upsilon}\kappa_1\left(\gamma_1-\left(\delta_{1,1}\mu_1-\kappa_1\right)\right) \\ -\frac{1}{\Upsilon}\kappa_2\delta_{1,2}\mu_2 \end{pmatrix} ,$$

where $M_2(t,\theta_0)$ *is a time-dependent matrix defined by*

$$M_2(t,\theta_0) = \begin{pmatrix} \theta_0^\top\left(Q_0-\gamma_1 I\right)^{-1} & 0 \\ 0 & \theta_0^\top\left(Q_0-\gamma_2 I\right)^{-1} \end{pmatrix} \tag{30}$$

$$\times \begin{pmatrix} \left(Q_0\right)^{-1}\left(\exp\left(Q_0 t\right)-I\right) - \frac{1}{\gamma_1}I\left(e^{\gamma_1 t}-1\right) & 0 \\ 0 & \left(Q_0\right)^{-1}\left(\exp\left(Q_0 t\right)-I\right) - \frac{1}{\gamma_2}I\left(e^{\gamma_2 t}-1\right) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1 & c_2 \end{pmatrix} .$$

Notice that the matrice Q_0 is not well conditioned as the sum of its column is the null vector. In theory, it is then not possible to invert it. However, the expression (28) may be calculated if we remember the definition of the matrix exponential. In this case, we calculate $Q_0^{-1}\left(\exp\left(Q_0 t\right)-I\right)$ by the following sum:

$$Q_0^{-1}\left(\exp\left(Q_0 t\right)-I\right) = Q_0^{-1}\left(I+\sum_{k=1}^{\infty}\frac{1}{k!}Q_0^k t^k - I\right)$$

$$= \sum_{k=1}^{\infty}\frac{1}{k!}Q_0^{k-1}t^k .$$

From the last proposition, we infer that the long term mean of the stock price is constant if $\gamma_1 < 0$ and $\gamma_2 < 0$. In this case, the asymptotic stock price is constant and detailed in the next corollary:

Corollary 3.8. *If γ_1 and γ_2 are strictly negative, the asymptotic expected price is equal to:*

$$\lim_{t\to\infty} \mathbb{E}\left(S_t|\mathcal{F}_0\right) = S_0 + \begin{pmatrix} -\alpha_1\mu_1 \\ \alpha_2\mu_2 \end{pmatrix}^\top V \lim_{t\to\infty} \begin{pmatrix} \int_0^t m_1\left(s,\theta_0\right) ds \\ \int_0^t m_2\left(s,\theta_0\right) ds \end{pmatrix}$$
$$+ \begin{pmatrix} -\alpha_1\mu_1 \\ \alpha_2\mu_2 \end{pmatrix}^\top V \begin{pmatrix} -\frac{1}{\gamma_1} & 0 \\ 0 & -\frac{1}{\gamma_2} \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_0^1 \\ \lambda_0^2 \end{pmatrix},$$

where the limits of integrals present in the first term are given by

$$\begin{aligned} \lim_{t\to\infty}\int_0^t m_1\left(s,\theta_0\right) ds &= e_1^\top M_2(\infty,\theta_0) \begin{pmatrix} \frac{1}{\Upsilon}\kappa_1\left(\left(\delta_{1,1}\mu_1-\kappa_1\right)-\gamma_2\right) \\ \frac{1}{\Upsilon}\kappa_2\delta_{1,2}\mu_2 \end{pmatrix} \\ \lim_{t\to\infty}\int_0^t m_2\left(s,\theta_0\right) ds &= e_2^\top M_2(\infty,\theta_0) \begin{pmatrix} \frac{1}{\Upsilon}\kappa_1\left(\gamma_1-\left(\delta_{1,1}\mu_1-\kappa_1\right)\right) \\ -\frac{1}{\Upsilon}\kappa_2\delta_{1,2}\mu_2 \end{pmatrix}, \end{aligned} \tag{31}$$

and where $M_2(\infty,\theta_0)$ is the constant matrix

$$M_2(\infty,\theta_0) = \begin{pmatrix} \theta_0^\top \left(Q_0-\gamma_1 I\right)^{-1}\left(\left(Q_0\right)^{-1}\left(\Pi-I\right)+\frac{1}{\gamma_1}I\right)c_1 & 0 \\ 0 & \theta_0^\top \left(Q_0-\gamma_2 I\right)^{-1}\left(\left(Q_0\right)^{-1}\left(\Pi-I\right)+\frac{1}{\gamma_2}I\right)c_2 \end{pmatrix}.$$

As mentioned in the introduction, we denote by $\psi_1(.)$ and $\psi_2(.)$ the moment generating of O_1 and O_2. The next proposition presents the Laplace transform of the number of jumps N^k, $k\in\{1,2\}$ which is the exponential of an affine function of the intensities. This result is very useful if we want to calculate numerically the first moments of orders counting processes.

Proposition 3.9. *For any $\omega\in\mathbb{R}$, the probability generating function for N_T^k, conditionally to $\mathcal{F}_t$, for $k=1,2$ with $T\geq t$ is given by*

$$\mathbb{E}\left(\omega^{N_T^k}\,|\,\mathcal{F}_t\right) = \omega^{N_t^k}\exp\left(A(t,T,\theta_t)+B_k(t,T)\lambda_t^k\right) \quad ,k\in\{1,2\}$$

where $B(t,T)$ is the solution of an ODE:

$$\begin{aligned} \frac{\partial}{\partial t}B_1 &= \kappa_1 B_1 - \left[1_{k=1}\omega\psi_1\left(B_1\delta_{1,1}+B_2\delta_{2,1}\right)-1\right], \\ \frac{\partial}{\partial t}B_2 &= \kappa_2 B_2 - \left[1_{k=2}\omega\psi_2\left(B_1\delta_{1,2}+B_2\delta_{2,2}\right)-1\right], \end{aligned} \tag{32}$$

with the terminal condition $B_k(T,T)=0$ for $k=1,2$. Let us define $\tilde{A}(t,T)=\left[e^{A(t,T,e_1)},\ ...,\ e^{A(t,T,e_l)}\right]^\top$. $\tilde{A}(t,T)$ is a vector, solution of the ODE system:

$$\frac{\partial\tilde{A}(t,T)}{\partial t} + \left(diag\left(\kappa_1 c_{1,t}\,B_1+\kappa_2 c_{2,t}\,B_2\right)+Q_0\right)\tilde{A}(t,T) = 0, \tag{33}$$

under the terminal boundary condition:

$$\tilde{A}(T,T) = 0_l\,.$$

where 0_l is the null vector of dimension l.

The proof is detailed in appendix. The next proposition presents the moment generating function (mgf) of S_t. The mgf may be inverted numerically by a discrete Fourier transform to retrieve the probability density function of S_t. We could eventually think to use this density to calibrate the model by log-likelihood maximization. In numerical applications, we instead opt for a MCMC algorithm which is detailed in Section 4.1.

Proposition 3.10. *For any $(\omega_1, \omega_2, \omega_3) \in \mathbb{C}_-{}^3$, the mgf of $\omega_1 S_T + \omega_2 \lambda_T^1 + \omega_3 \lambda_T^2$, conditionally to $\mathcal{F}_t$, for $T \geq t$, is given by the following expression*

$$\mathbb{E}\left(e^{\omega_1 S_T + \omega_2 \lambda_T^1 + \omega_3 \lambda_T^2} \mid \mathcal{F}_t\right) = \exp\left(\omega_1 S_t + A(t,T,\theta_t) + B_1(t,T)\lambda_t^1 + B_2(t,T)\lambda_t^2\right),$$

where $B_1(t,T)$ and $B_2(t,T)$ are functions of time, solutions of the ODE:

$$\begin{aligned} \frac{\partial}{\partial t} B_1 &= \kappa_1 B_1 - \omega_1 \alpha_1 \mu_1 - \left[\psi_1\left(B_1 \delta_{1,1} + B_2 \delta_{2,1} + C_1\right) - 1\right] \\ \frac{\partial}{\partial t} B_2 &= \kappa_2 B_2 + \omega_1 \alpha_2 \mu_2 - \left[\psi_2\left(B_1 \delta_{1,2} + B_2 \delta_{2,2} + C_2\right) - 1\right], \end{aligned} \tag{34}$$

with the terminal condition $B_1(T,T) = \omega_2$ and $B_2(T,T) = \omega_3$. And where

$$\tilde{A}(t,T) = \left[e^{A(t,T,e_1)}, \ldots, e^{A(t,T,e_l)}\right]^\top$$

is a vector of functions, solution of the ODE system:

$$\frac{\partial \tilde{A}(t,T)}{\partial t} + \left(diag\left(\kappa_1 c_{1,t}\, B_1 + \kappa_2 c_{2,t}\, B_2\right) + Q_0\right) \tilde{A}(t,T) = 0.$$

under the terminal boundary condition:

$$\tilde{A}(T,T) = 0_l\,.$$

where 0_l is the null vector of dimension l.

The proof of this result being similar to the one of proposition 3.9, we do not provide it. The functions $B_1(t,T)$ and $B_2(t,T)$ in proposition 3.9 do not admit any simple analytical expression. However, they can be reformulated as solution of a non-linear system of equations. Furthermore, we can find the domain of $\mathbb{R}$, on which these functions are defined as stated in the next proposition.

Proposition 3.11. *for $k = 1, 2$, let us define*

$$\beta_k(\omega_1) = (-1)^k \omega_1 \alpha_k \mu_k + 1\,, \tag{35}$$

and functions $F_{\omega_1}^1(x,y)\,:\,\mathbb{R}^2 \to \mathbb{R}^+$,

$$\begin{aligned} F_{\omega_1}^1(x,y) &:= \int_{\omega_2}^{x} \frac{du_1}{-\kappa_1 u_1 + \psi_1\left(u_1 \delta_{1,1} + y \delta_{2,1} + C_1\right) - \beta_1(\omega_1)}, \\ F_{\omega_1}^2(x,y) &:= \int_{\omega_3}^{y} \frac{du_2}{-\kappa_2 u_2 + \psi_2\left(x \delta_{1,2} + u_2 \delta_{2,2} + C_2\right) - \beta_2(\omega_1)}. \end{aligned} \tag{36}$$

If $\left(F^1_{\omega_1}\right)^{-1}(\tau\,|\,y)$ *and* $\left(F^2_{\omega_1}\right)^{-1}(\tau\,|\,x)$ *are respectively the inverse functions of* $F^1_{\omega_1}(.,y)$ *and* $F^2_{\omega_1}(x,.)$, *then the functions* $B_1(t,T)$ *and* $B_2(t,T)$ *solution of ODEs (34)*

$$\begin{cases} B_1(t,T) = \left(F^1_{\omega_1}\right)^{-1}(T-t\,|\,B_2(t,T)) \\ B_2(t,T) = \left(F^2_{\omega_1}\right)^{-1}(T-t\,|\,B_1(t,T)) \end{cases}$$

And for $k \in \{1,2\}$, $B_k \in [\omega k+1, u^*_k)$ *or* $B_k \in [u^*_k, \omega k+1)$ *where* (u^*_1, u^*_2) *is the unique solution of the system:*

$$\psi_k\left(u_1\delta_{1,k} + u_2\delta_{2,k} + C_k\right) = \beta_k(\omega_1) + \kappa_k u_k$$

In numerical applications, we prefer to solve numerically ODE's (34) instead of inverting functions $F^1_{\omega_1}$ and $F^2_{\omega_1}$, which reveals hard to numerically invert in practice.

4 Estimation of parameters

Given that economic regimes and jump intensities are not directly observable, the estimation of SMM parameters is challenging. On the other hand, the statistical distribution of prices does not admit a closed form expression. It is then not possible to infer parameters by log-likelihood maximization. Instead, we use a Particle Monte Carlo Markov Chain (PMCMC) method to fit the SMM to a time serie. The PMCMC algorithm is based on a particle filter that evaluates the log-likelihood by simulations. The next paragraph details this filter.

4.1 A Particle filter

The Markov chain θ_t and intensities of jumps λ^1_t, λ^2_t, are hidden state variables. We use then a sequential Monte-Carlo (SMC) method, also called particle filter, to guess their sample paths. This Bayesian technique is combined later with a Monte-Carlo Markov Chain to fit the SMM, but for the moment, we assume that parameters are known. The procedure is based on a discrete versions of equations (4) defining prices and (11) that drives the jumps arrival intensities. We denote by Δ the length of the time interval. The *ex ante* variation of prices (over the period Δ) at time $t_j = j\Delta$, defined by $X_j = S_{(j+1)\Delta} - S_{j\Delta}$, then satisfies the following equation in discrete time

$$X_j = \alpha_2 \mathbf{\Delta} L^2_j - \alpha_1 \mathbf{\Delta} L^1_j\,, \tag{37}$$

where $\mathbf{\Delta} L^i_j = \sum_{u=N^i_{j\Delta}}^{N^i_{(j+1)\Delta}} O^i_u$ for $i = 1,2$ is the sum of buy-sell orders. In the discretized framework, $N^i_{(j+1)\Delta} - N^i_{j\Delta}$ are Poisson random variables with a constant intensity $\lambda^i_j\Delta$, over Δ . The economic regime is assumed to remain unchanged over the time interval Δ and the value of θ_t for $t \in [j\Delta\,,\,(j+1)\Delta]$ is denoted by θ_j. The mean reversion levels of λ^1_j and λ^2_j are constant over the j^{th} interval of time and equal to $c_{i,j} = c_i^\top\theta_j$ for $i = 1,2$.

The Euler approximation of equations (11) provides the discrete dynamics of latent processes $\lambda^i = \left(\lambda^i_t\right)_t$:

$$\lambda^i_{j+1} = \lambda^i_j + \quad \kappa_i(c_{i,j} - \lambda^i_j)\Delta + \delta_{i,1}\Delta L^1_j + \delta_{i,2}\Delta L^2_j \qquad i = 1,2\,. \tag{38}$$

The second latent process carries the information about the economic regime. We denote by $(\theta_j)_{j\in\mathbb{N}}$ the discrete Markov chain approximating θ_t. This chain has a matrix of transition probabilities denoted $P_\Delta = \exp(Q_0\Delta)$ and the transition random measure $K(.)$ such that $\theta_{j+1} = \int_{\theta\in E} K(\theta_j, d\theta)$. Remember that at this stage, the model parameters are assumed to be known. A particle at time t_j is a triplet denoted by $v_j = (\lambda_j^1, \lambda_j^2, \theta_j)$ that contains information about the economic regime and intensities. The model admits a useful state-space representation, where the equation (37) provides a measurement equation or system (the 'space') that defines the relationship between variations of prices and hidden state variables. The particle v_j helps to find the transition system (the 'state') that describes the dynamics of state variables.

In the remainder of the paper, we denote by $\{x_1, x_2, ..., x_n\}$, the sample of observed variations of stock prices, realisations of X_j for $j = 1, ..., n$. Conditionally to information contained in v_j, the probability density function (pdf) of price variations at time j $p(x_j|v_j)$ is given by

$$\begin{aligned} p\left(x_j \mid v_j\right) \;=\; & \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\Big[P\left(N^1_{(j+1)\Delta} - N^1_{j\Delta} = k_1 \mid \lambda_j^1\right) \\ & \times P\left(N^2_{(j+1)\Delta} - N^2_{j\Delta} = k_2 \mid \lambda_j^2\right) \times f_{k_1,k_2}(x_j) \\ & + P\left(N^1_{(j+1)\Delta} - N^1_{j\Delta} = 0 \mid \lambda_j^1\right) \\ & \times P\left(N^2_{(j+1)\Delta} - N^2_{j\Delta} = 0 \mid \lambda_j^2\right) \mathbf{1}_{\{x_j=0\}}\Big] , \end{aligned} \tag{39}$$

where $f_{k_1,k_2}(.)$ is the convoluted law of the sum of random variables: $\alpha_2 \sum_{i=1}^{k_2} O_i^2 - \alpha_1 \sum_{i=1}^{k_1} O_i^1$. Given that the interval Δ between two successive observations is small, the probability of observing more than one or two jumps is negligible. The sum in equation (39) may then be limited to a few terms in order to reduce the computation time. In numerical applications orders are assumed distributed as normal random variables and α_1, α_2 are set to one. The sum $\alpha_2 \sum_{i=1}^{k_2} O_i^2 - \alpha_1 \sum_{i=1}^{k_1} O_i^1$ is then Gaussian with a mean and a standard deviation respectively equal to $(k_2\mu_2 - k_1\mu_1)$ and $\sqrt{k_2\sigma_2^2 + k_1\sigma_1^2}$. Here σ_1^2 and σ_2^2 are the variance of O^1 and O^2.

On the other hand, it is also possible to simulate the transition density $p(v_{j+1} \mid v_j)$ with equations (38) and $\theta_{j+1} = \int_{\theta\in E} K(\theta_j, d\theta)$. The density of the intial particle v_0 is denoted by $p(v_0)$ and the posterior distribution of v_j given observations till time t_j, is denoted by $p(v_j \mid x_{1:j})$. Using the Bayes' rule, the posterior distribution is developped as follows

$$p(v_j \mid x_{1:j}) = \frac{p(x_{1:j}\,,\, v_j\,)}{p(x_{1:j})}, \tag{40}$$

and the denominator satisfies the equality:

$$p(x_{1:j}) = p(x_{1:j-1}, x_j) = p(x_j \mid x_{1:j-1})p(x_{1:j-1})\,.$$

Given that the numerator of equation (40) is equal to

$$p(x_{1:j}\,,\, v_j\,) = p(x_j \mid v_j)p(v_j \mid x_{1:j-1})p(x_{1:j-1}),$$

the expression for the posterior distribution is rewritten as:

$$p(v_j \mid x_{1:j}) = \frac{p(x_j \mid v_j)}{\int p(x_j|v_j)p(v_j \mid x_{1:j-1})dv_j}p(v_j \mid x_{1:j-1})\,, \tag{41}$$

where

$$p(v_j \mid x_{1:j-1}) = \int p(v_j \mid v_{j-1})p(v_{j-1} \mid x_{1:j-1})dv_{j-1}\,. \tag{42}$$

The calculation of $p(\lambda_j^1, \lambda_j^2, \theta_j \,|\, x_{1:j})$ is done in two steps. The first one is a prediction step in which we estimate $p(v_j \,|\, x_{1:j-1})$ by the relation (42). In the correction step, we approach the probabilities $p(v_j \,| x_{1:j})$ using the equation (41). In practice, the integral in the prediction step is replaced by a Monte Carlo simulation, of M particles, denoted by $v_j^{(k)} = (\lambda_j^{1\,(k)}, \lambda_j^{2\,(k)}, \theta_j^{(k)})$ for $k = 1, \ldots, M$. The structure of the particle filter algorithm is the following:

Particle filter algorithm

1. **Initial step**: draw M values of $v_0^{(k)}$ for $k = 1, \ldots, M$, from an initial distribution $p(v_0)$

2. For $j = 1 \,:\, T$
   Prediction step: draw a sample of $\Delta L_j^{1\,(k)}$, $\Delta L_j^{2\,(k)}$ and $\theta_j^{(k)}$ and update $\lambda_j^{1\,(k)}$, $\lambda_j^{2\,(k)}$, $c_j^{1\,(k)}$, $c_j^{2\,(k)}$ using the relations (38) $c_j^{1\,(k)} = c_1^\top \theta_j^{(k)}$, and $c_j^{2\,(k)} = c_2^\top \theta_j^{(k)}$.

   Correction step: the particle $v_j^{(k)}$ has a probability of $w_j^{(k)} = \frac{p(x_j \,|\, v_j)}{\sum_{k=1:M} p(x_j \,|\, v_j^{(k)})}$ where $p(x_j \,|\, v_j^{(k)})$ is distributed according to the mixture distribution of equation (39).

   Resampling step: resample with replacement M particles according to the importance weights $w_j^{(k)}$. The new importance weights are set to $w_j^{(k)} = \frac{1}{M}$.

Finally, the filtered intensities for the period j is computed as the sum of particles, weighted by their probabilities of occurrence:

$$\mathbb{E}\left(\lambda_j^1 \,|\, \mathcal{G}_T\right) = \sum_{i=1:M} \lambda_j^{1\,(i)}\, w_j^{(k)} \qquad \mathbb{E}\left(\lambda_j^2 \,|\, \mathcal{G}_T\right) = \sum_{i=1:M} \lambda_j^{2\,(i)}\, w_j^{(k)}\,,$$

whereas the log-likelihood is approached as follows:

$$\begin{aligned} \log L(\Theta) &= \sum_{j=1}^{T} \log\, p\left(x_j \,|\, x_{j-1}\right) \\ &= \sum_{j=1}^{T} \log \int p(x_j \,|\, v_j) p(v_j | v_{j-1}) dv_j \\ &= \sum_{j=1}^{T} \log \left(\frac{1}{M} \sum_{k=1}^{M} p(x_j \,|\, v_j^{(k)}) \right) . \end{aligned} \tag{43}$$

However, the estimator of the likelihood is not continuous as a function of parameters because it is based on simulations. Fitting parameters by log-likelihood maximization is then inefficient. This observation justifies working with a Monte-Carlo Markov Chain algorithm.

4.1.1 Application on simulated data

To conclude this section, we test the performance of the SMC filter with a simulated data-set. We first simulate a daily sample path of a SMM, with three economic regimes and over a period of ten years. The parameters used for this simulation[1] are reported in table 1. The first state corresponds

[1]Chosen parameters are in the same range of values as real estimates reported in Section 4.2. In order to clearly vizualize changes of regimes, the gap between mean reversion levels in each regimes is increased. For the same reason, we have also modified transition probabilities in order to observe a sufficient number of changes of regime during the simulation.

to a period of economic recession: negative average return, high volatility and frequency of jumps. The third regime represents a period of economic growth: positive expected return, low volatility and frequency of jumps. The second state is an intermediate conjuncture, close to economic stagnation. The one year matrix of transition probabilities used for this exercise is presented in table 2.

$c_{1,1}$	10	$c_{2,1}$	20	κ_1	56	κ_2	50
$c_{1,2}$	35	$c_{2,2}$	70	α_1	1	α_2	1
$c_{1,3}$	50	$c_{2,3}$	100	μ_1	10	μ_2	10
$\delta_{1,1}$	1	$\delta_{2,1}$	1	σ_1	5	σ_2	3
$\delta_{1,2}$	1	$\delta_{2,2}$	1	λ_0^1	12	λ_0^2	17

Table 1: This table reports the parameters used for the simulation of a daily sample path of the SMM, with three regimes.

$(p_{ij}(0,1))_{i,j=1,2,3}$	state 1	state 2	state 3
state 1	0.60	0.20	0.20
state 2	0.20	0.60	0.20
state 3	0.25	0.25	0.50

Table 2: This table presents the one year matrix of transition of θ_t, used for the simulation of a daily sample path of the SMM, with three regimes.

After simulation of a sample path, we run the SMC filter with 500 particles. The graphs of figure 1 compare simulated and filtered intensities of jumps and economic regimes. They confirm the efficiency of the SMC algorithm. This filter is combined in a next section to a Monte Carlo Markov Chain (MCMC) algorithm to estimate the SSEJD. But before, we introduce an approached estimation method that is used to initialize the MCMC algorithm.

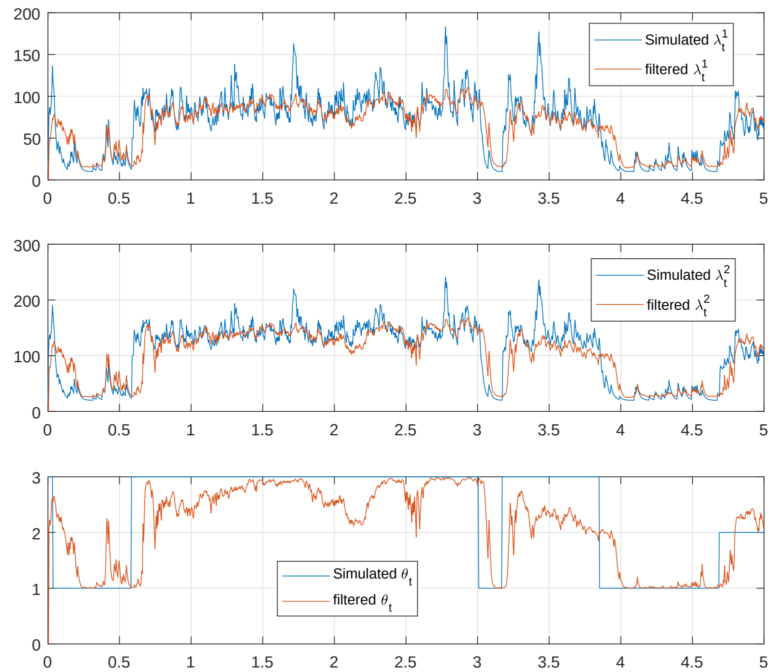

Figure 1: This graph shows simulated and filtered sample paths of λ_t^1, λ_t^2 and θ_t.

4.2 Calibration by Particle Monte Carlo Markov Chain

When dealing with a non-Gaussian and nonlinear specification, simulation-based methods offer strong advantages over the alternative approaches. In this paper, we employ a Particle Markov Chain Monte Carlo method (PMCMC) to fit the SMM to a time-serie. We refer to Doucet et al. (2000) for a review of other simulation-based methods. The set of parameters is denoted by Ξ and serves us as index for the probability distribution function. We adopt a Bayesian approach to estimate Ξ by computing the parameters posterior distribution

$$\pi(\Xi) \quad = \quad p(\Xi \,|\, x_{1:T}) = \frac{p(\Xi)p(x_{1:T} \,|\, \Xi)}{\int p(\Xi')p(x_{1:T} \,|\, \Xi')d\Xi'} \,, \tag{44}$$

where $p(\Xi)$ and $p(x_{1...T}|\Xi)$ denotes respectively the parameters prior distribution and the likelihood of the data. The density $\pi(\Xi)$ is built by the PMCMC method that generates a sample from $\pi(\Xi)$ by creating a Markov chain with the same stationary distribution as the parameters posterior one. Once that the Markov chain has reached stationarity after a transient phase, called "burn-in" period, samples from the posterior distribution can then be simulated. Standard MCMC algorithm requires a point-wise estimate of $p(x_{1...T}|\Xi)$, that is not available in our model. Instead, $p(x_{1...T}|\Xi)$ is approached by its estimate computed with a particle filter.

The construction of the Markov chain consists of two steps, iteratively repeated. At the beginning of the k^{th} iteration, we propose a candidate parameter Ξ' from a proposal distribution $q(\Xi'|\Xi^{(k-1)})$ given the previous state of the Markov chain, noted $\Xi^{(k-1)}$. The proposal distribution has a support that covers the target distribution. In the second step, we determine if we update the state by Ξ'.

For this purpose, the acceptance probability is computed as follows

$$\varepsilon(\Xi', \Xi^{(k-1)}) = \min\left\{1, \frac{\pi(\Xi')}{\pi(\Xi^{(k-1)})} \frac{q\left(\Xi^{(k-1)} \,|\, \Xi'\right)}{q\left(\Xi' \,|\, \Xi^{(k-1)}\right)}\right\}. \tag{45}$$

This determines the probability that we assign the candidate parameter as the next state of the Markov chain, $\Xi' \to \Xi^{(k)}$. Intuitively, if we disregard the influence of the proposal q, a candidate is accepted if it increases the posterior likelihood $\pi(\Xi') > \pi(\Xi^{(k-1)})$. The presence of $q(.)$ in equation (45) allows a small decrease in the posterior likelihood, so as to explore the entire posterior.

The resulting K samples $\Xi^{(1:K)}$ (after the burn in period) serve next to build the empirical distribution of $\pi(\Xi)$, which is defined by

$$\hat{\pi}(\Xi) = \frac{1}{K}\sum_{k=1}^{K} \delta_{\Xi^{(k)}}(d\Xi),$$

where $\delta_{\Xi^{(k)}}(d\Xi)$ are the Dirac atoms located at $\Xi = \Xi^{(k)}$, with equal weights. The expected parameters with respect to the posterior distribution of parameters is then approached as follows

$$\mathbb{E}\left(\Xi | x_{1:T}\right) \approx \frac{1}{K}\sum_{k=1:K} \hat{\pi}(\Xi^{(k)})\,\Xi^{(k)}.$$

In numerical applications, the transition distribution $q(\Xi' \,|\, \Xi^{(k-1)})$ is assumed Normal, $\mathcal{N}(\Xi' \,|\, \Xi^{(k-1)}, \sigma_q)$. As this distribution is symmetric, $q\left(\Xi^{(k-1)} \,|\, \Xi'\right) = q\left(\Xi' \,|\, \Xi^{(k-1)}\right)$, the acceptance probability simplifies to

$$\begin{aligned}\varepsilon(\Xi', \Xi^{(k-1)}) &= \min\left\{1, \frac{\pi(\Xi')}{\pi(\Xi^{(k-1)})}\right\}, \\ &= \min\left\{1, \frac{p(\Xi')p(x_{1:T} \,|\, \Xi')}{p(\Xi^{(k-1)})p(x_{1:T} \,|\, \Xi^{(k-1)})}\right\}.\end{aligned}$$

We use the PMCMC algorithm to calibrate the SMM model. We test the calibration algorithm with daily data of the S&P 500, from February 2010 to February 2017 (1763 observations). We use this dataset to compare the classic approach without modulation of parameters, to models with 2 and 3 regimes. The PMCMC algorithm is applied to the set of parameters

$$\Xi = \{\bar{c}, \kappa_1, \kappa_2, \mu_1, \mu_2, \sigma_1, \sigma_2, (\delta_{i,j})_{i,j=1,2}, (q_{i,j})_{i,j=1,2}\}$$

that counts respectively 12, 16 and 22 parameters with 1, 2 and 3 regimes. Gatumel and Ielpo (2014) reject the hypothesis that two regimes are enough to capture asset returns evolutions for many securities. Their empirical results point out that between two and three regimes are required to capture the features of asset's distribution.

The filter runs with 500 particles and we perform 5000 iterations for the PMCMC procedure. We obtain acceptance rates of 36.41% and 41.16% for models with respectively 2 and 3 regimes. The convergence is checked by analyzing the log-likelihood, which is stable for both models after a burn in period of 2500 iterations. The average log-likelihoods over the last 2500 runs are reported in table 3. We also report the Akaike and Bayesian information criterions. These figures clearly confirm that switching models outperform the classic microstructure model with a single regime.

	1D	2D	3D
Log-likelihood	-7242	-7119	-7049
AIC	7266	7151	7093
BIC	14574	14358	14264

Table 3: Log-likelihood, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC)

Parameters estimated by the PMCMC algorithm are reported in tables 4, 5 and 6. The speeds of mean reversion (κ_1, κ_2) are comparable for all models. In the 1D and 3D SMM, the parameters of mutual-excitation (δ_{12}, δ_{21}) are less important than these of self-excitation (δ_{11}, δ_{22}). For the 2D model, the situation is different and mutual excitation is more pronounced than the self-excitation. Averages and standard deviations of orders are similar whatever the model. Figures reported in table 5, reveals that the probabilities of staying in the same state over a period of one year are respectively around 56% and 36% for the 2D and 3D SMMs.

Table 6 compares the reversion levels of intensities in each regime. In the one dimension model, these levels for the supply and demand are comparable. In the 2D SMM, $c_{1,1}$, $c_{1,2}$ are respectively higher and lower than $c_{2,1}$, $c_{2,2}$. This means that the market receives more bid than ask orders in the first regime, and more ask than bid orders in the second regime. As bid and ask orders respectively drive down and up the stock price, the first regime is then assimilated to the conditions of a bear market. Whereas the second regime corresponds to a period of economic growth.

	1D		2D		3D	
	Estimate	St.dev.	Estimate	St.dev.	Estimate	St.dev.
κ_1	21.316	2.790	25.802	5.283	43.187	4.765
κ_2	38.238	4.678	33.828	6.010	38.408	4.289
δ_{11}	2.881	0.347	0.891	0.367	8.033	0.915
δ_{12}	0.2580	0.223	5.136	1.035	0.190	0.250
δ_{21}	0.132	0.152	4.512	0.532	0.184	0.188
δ_{22}	7.951	0.615	0.360	0.316	6.962	0.641
μ_1	6.708	0.403	5.220	0.255	4.880	0.303
μ_2	4.065	0.362	4.530	0.424	4.986	0.340
σ_1	2.133	0.712	3.726	0.489	4.742	1.155
σ_2	5.723	0.430	3.962	0.443	3.271	0.409

Table 4: This table reports parameters independent from θ_t, for the 1D, 2D and 3D MSM models, (averages and standard deviations over the last 2500 simulations).

Transition matrix of probabilities	
2D	$P = \begin{pmatrix} 0.572 & 0.428 \\ 0.433 & 0.567 \end{pmatrix}$
3D	$P = \begin{pmatrix} 0.369 & 0.296 & 0.335 \\ 0.333 & 0.333 & 0.334 \\ 0.333 & 0.299 & 0.368 \end{pmatrix}$

Table 5: Matrix of transition probabilities, for the SMM models with 2 and 3 regimes.

		Estimate	St.dev.		Estimate	St.dev.
1D	c_1	27.334	6.5064	c_2	23.203	3.0661
2D	$c_{1,1}$	48.858	9.741	$c_{2,1}$	20.896	4.3289
	$c_{1,2}$	21.43	5.0021	$c_{2,2}$	32.234	7.087
3D	$c_{1,1}$	27.035	6.2039	$c_{2,2}$	2.4002	1.5993
	$c_{1,2}$	11.852	2.4269	$c_{2,2}$	27.471	7.2078
	$c_{1,3}$	15.459	3.5566	$c_{2,3}$	7.6532	4.9848

Table 6: Mean reversion levels of intensities in each regime for the three tested models.

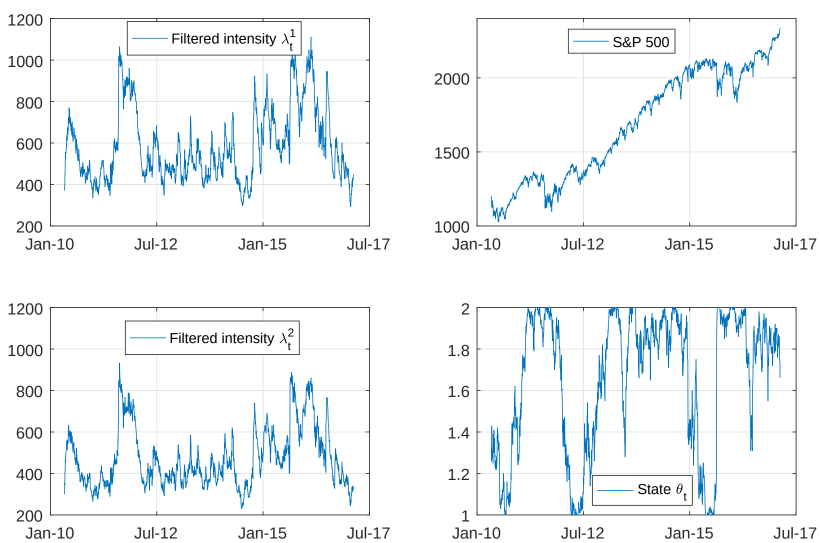

Figure 2: This graph shows filtered sample paths of λ_t^1, λ_t^2 and θ_t for the model with 2 regimes. The upper right plot presents the history of the S&P 500 from 2010 to 2017.

This interpretation is confirmed by the right graphs of figure 2 that display the evolution of the

S&P index 500 and filtered states. We observe a switch toward the first regime when the index drops. We draw the same conclusion for the 3D SMM, in which states one and two correspond respectively to bear and bull markets. Whereas the third regime is an intermediate state in which stock prices stagnate. The filtered regime informs us about the mood of markets. We may then imagine to use this information to define a dynamic trading strategy as proposed in Hainaut and MacGilchrist (2012). Finally, the two left graphs of figure 2 exhibit the filtered sample path of λ_t^1 and λ_t^2. We observe that intensities reach their highest level when the S&P 500 falls. Monitoring these intensities could then be used by regulators to measure the stocks market stress.

5 Conclusions

This article proposes a new microstructure model for stock prices with regime shifts and mutual-excitation in the dynamic of orders arrivals. In this approach, called the switching microstructure model (SMM), the intensities of orders counting processes revert to a mean level that is modulated by a hidden Markov chain. This chain determines the direction of the market trend and the trading behaviour. In the first part of this work, we study the mathematical properties of the SMM. We show that the SMM presents a sufficient degree of analytical tractability for most of applications. The rest of the article focuses on the estimation of parameters.

The probability density function of prices does not have a closed form expression and increments of prices are not identically, independently distributed. Furthermore, prices depend upon three hidden state variables: the two mutually excited intensities of orders counting processes and the Markov chain. It is then not possible to estimate the SMM parameters by log-likelihood maximization. Instead, we develop a new sequential Monte Carlo algorithm to filter hidden processes, that is combined with a Markov Chain Monte Carlo (MCMC) procedure to estimate parameters.

The model is next fitted to daily returns of the S&P 500 stock index. This exercise reveals that the SMM with two and three regimes have a better explanatory power than a model without regime shift. Each state of the hidden Markov chain clearly corresponds to a particular trading trend. In the 3 states model, two regimes respectively correspond to a bear and a bull market whereas stock prices stagnate in the third regime. Filtering the evolution of the Markov chain can then help traders to adjust their positions to take advantage of market conditions. The filtered intensities of orders counting processes are also excellent indicators of markets stress.

Appendix

Proof of lemma 2.1 To prove this relation, we differentiate the expression of λ_t^i to retrieve its dynamic:

$$\begin{aligned} d\lambda_t^i &= \kappa_i c_{i,t} - \kappa_i \left(\lambda_0^i - \kappa_i \int_0^t e^{\kappa_i (s-t)} \left(\lambda_0^i - c_{i,s} \right) ds + \right. \\ & \left. \int_0^t \delta_{i,1} e^{\kappa_i (s-t)} dL_s^1 dt + \int_0^t \delta_{i,2} e^{\kappa_i (s-t)} dL_s^2 \right) + \delta_{i,1} dL_t^1 + \delta_{i,2} dL_t^2 \\ &= \kappa_i (c_{i,t} - \lambda_t^i) dt + \delta_{i,1} dL_t^1 + \delta_{i,2} dL_t^2 \qquad i = 1, 2. \end{aligned}$$

To prove the positivity, we first remind that $\int_0^t \delta_{i,1} e^{\kappa_i(s-t)} dL_s^1$ and $\int_0^t \delta_{i,2} e^{\kappa_i(s-t)} dL_s^2$ are positive by construction. According to equation (12), the process λ_t^i admit the following lower bound:

$$\lambda_t^i \quad > \quad \lambda_0^i + \left(\min\left(c_i\right) - \lambda_0^i\right) \kappa_i \int_0^t e^{\kappa_i(s-t)} ds \,. \tag{46}$$

Given that $\kappa_i \int_0^t e^{\kappa_i(s-t)} ds = \left(1 - e^{-\kappa_i t}\right) > 0$, we conclude that

$$\lambda_t^i \quad > \quad \lambda_0^i e^{-\kappa_i t} + \min\left(c_i\right)\left(1 - e^{-\kappa_i t}\right) > 0\,.$$

■

Proof of proposition 3.1. As $\mathcal{F}_s \subset \mathcal{F}_s \vee \mathcal{G}_t$, using nested expectations leads to the following expression for the expected intensity:

$$\mathbb{E}(\lambda_t^i | \mathcal{F}_s) \quad = \quad \mathbb{E}\left(\mathbb{E}\left(\lambda_t^i | \mathcal{F}_s \vee \mathcal{G}_t\right) | \mathcal{F}_s\right)\,.$$

If we remember the expression (13) of the intensity, using the Fubini's theorem leads to the following expression for the expectation of λ_t^i, conditionally to the augmented filtration $\mathcal{F}_s \vee \mathcal{G}_t$:

$$\begin{aligned} \mathbb{E}\left(\lambda_t^i | \mathcal{F}_s \vee \mathcal{G}_t\right) \quad &= \quad \lambda_s^i - \kappa_i \int_s^t e^{\kappa_i(u-t)} \left(\lambda_s^i - c_{i,u}\right) du \\ &+ \int_s^t \delta_{i,1} e^{\kappa_i(u-t)} \mathbb{E}\left(dL_u^1 | \mathcal{F}_s \vee \mathcal{G}_t\right) + \int_s^t \delta_{i,2} e^{\kappa_i(u-t)} \mathbb{E}\left(dL_u^2 | \mathcal{F}_s \vee \mathcal{G}_t\right)\,. \end{aligned} \tag{47}$$

Using the same approach as in Errais et al. (2010), dL_u^i is rewritten as follows:

$$dL_u^i = \begin{cases} O^i & if\; dN_u^1 = 1 \\ 0 & otherwise \end{cases}\,.$$

The order size O_i being independent from all processes and then from $\mathcal{F}_s \vee \mathcal{G}_t$, we infer that

$$\begin{aligned} \mathbb{E}\left(dL_u^i | \mathcal{F}_s \vee \mathcal{G}_t\right) &= \mathbb{E}\left(O^i\right) \times \mathbb{E}\left(dN_u^1 \,|\, \mathcal{F}_s \vee \mathcal{G}_t\right) \\ &= \mu_i \times \mathbb{E}\left(dN_u^1 \,|\, \mathcal{F}_s \vee \mathcal{G}_u\right)\,. \end{aligned}$$

Using nested expectations and conditioning with respect to the sample path of λ_u^1 contained in the subfiltration $\mathcal{H}_u$ of $\mathcal{F}_u$ leads to equality for $u \leq t$

$$\mathbb{E}\left(dL_u^i | \mathcal{F}_s \vee \mathcal{G}_t\right) = \mu_i \times \mathbb{E}\left(\mathbb{E}\left(dN_u^1 \,|\, \mathcal{F}_s \vee \mathcal{G}_u \vee \mathcal{H}_u\right) \,|\, \mathcal{F}_s \vee \mathcal{G}_u\right)\,.$$

Conditionally to the sample path of λ_u^1, N_u^1 is a non-homogeneous Poisson process,

$$\mathbb{E}\left(dN_u^1 \,|\, \mathcal{F}_s \vee \mathcal{G}_u \vee \mathcal{H}_u\right) = \lambda_{u-}^i$$

therefore, we infer that

$$\mathbb{E}\left(dL_u^i | \mathcal{F}_s \vee \mathcal{G}_t\right) = \mu_i \times \mathbb{E}\left(\lambda_{u-}^i \,|\, \mathcal{F}_s \vee \mathcal{G}_u\right) du \quad \forall\, u \leq t,$$

If we derive equation (47) with respect to time, we find that $\mathbb{E}\left(\lambda_t^i | \mathcal{F}_s \vee \mathcal{G}_t\right)$ is solution of an ordinary differential equation (ODE):

$$\begin{aligned} \frac{\partial}{\partial t} \mathbb{E}\left(\lambda_t^i | \mathcal{F}_s \vee \mathcal{G}_t\right) &= -\kappa_i \left(\lambda_s^i - c_{i,t}\right) + \kappa_i^2 \int_s^t e^{\kappa_i(u-t)} \left(\lambda_s - c_{i,u}\right) du \\ &+ \delta_{i,1}\mu_1 \mathbb{E}\left(\lambda_t^1 | \mathcal{F}_s \vee \mathcal{G}_t\right) - \kappa_i \delta_{i,1}\mu_1 \int_s^t e^{-\kappa_i(t-u)} \mathbb{E}\left(\lambda_{u-}^1 \,|\, \mathcal{F}_s \vee \mathcal{G}_t\right) du \\ &+ \delta_{i,2}\mu_2 \mathbb{E}\left(\lambda_t^2 | \mathcal{F}_s \vee \mathcal{G}_t\right) - \kappa_i \delta_{i,2}\mu_2 \int_s^t e^{-\kappa_i(t-u)} \mathbb{E}\left(\lambda_{u-}^2 \,|\, \mathcal{F}_s \vee \mathcal{G}_t\right) du\,. \end{aligned}$$

Using equation ((47)), allows us to rewrite these ODE's as follows:

$$\begin{pmatrix} \frac{\partial}{\partial t}\mathbb{E}\left(\lambda_t^1|\mathcal{F}_s\vee\mathcal{G}_t\right) \\ \frac{\partial}{\partial t}\mathbb{E}\left(\lambda_t^2|\mathcal{F}_s\vee\mathcal{G}_t\right) \end{pmatrix} = \begin{pmatrix} \kappa_1 c_{1,t} \\ \kappa_2 c_{2,t} \end{pmatrix} + \begin{pmatrix} \delta_{1,1}\mu_1-\kappa_1 & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & \delta_{2,2}\mu_2-\kappa_2 \end{pmatrix} \begin{pmatrix} \mathbb{E}\left(\lambda_t^1|\mathcal{F}_s\vee\mathcal{G}_t\right) \\ \mathbb{E}\left(\lambda_t^2|\mathcal{F}_s\vee\mathcal{G}_t\right) \end{pmatrix} \quad (48)$$

Solving this system of equation requires to determine eigenvalues γ and eigenvectors (v_1, v_2) of the matrix present in the right term of this system:

$$\begin{pmatrix} (\delta_{1,1}\mu_1-\kappa_1) & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & (\delta_{2,2}\mu_2-\kappa_2) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \gamma \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

We know that eigenvalues cancel the determinant of the following matrix:

$$\det\begin{pmatrix} (\delta_{1,1}\mu_1-\kappa_1)-\gamma & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & (\delta_{2,2}\mu_2-\kappa_2)-\gamma \end{pmatrix} = 0\,,$$

and are solutions of the second order equation:

$$\gamma^2-\gamma\left((\delta_{1,1}\mu_1-\kappa_1)+(\delta_{2,2}\mu_2-\kappa_2)\right)+(\delta_{1,1}\mu_1-\kappa_1)(\delta_{2,2}\mu_2-\kappa_2)-\delta_{1,2}\delta_{2,1}\mu_1\mu_2 = 0.$$

Roots of the last equation are γ_1 and γ_2, as defined by the equation (14). One way to find an eigenvector is to note that it must be orthogonal to each rows of the matrix:

$$\begin{pmatrix} (\delta_{1,1}\mu_1-\kappa_1)-\gamma & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & (\delta_{2,2}\mu_2-\kappa_2)-\gamma \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

then necessary,

$$\begin{pmatrix} v_1^i \\ v_2^i \end{pmatrix} = \begin{pmatrix} -\delta_{1,2}\mu_2 \\ (\delta_{1,1}\mu_1-\kappa_1)-\gamma_i \end{pmatrix} \quad for\ i=1,2.$$

If we note $D := diag(\gamma_1, \gamma_2)$, the matrix in the right term of equation (48) admits the decomposition:

$$\begin{pmatrix} \delta_{1,1}\mu_1-\kappa_1 & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & \delta_{2,2}\mu_2-\kappa_2 \end{pmatrix} = VDV^{-1},$$

where V is the matrix of eigenvectors, as defined in equation (16). Its determinant, Υ, and its inverse are respectively provided by equations (18) and (17). If two new variables are defined as follows:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = V^{-1}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}.$$

The system (48) is decoupled into two independent ODEs:

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = V^{-1}\begin{pmatrix} \kappa_1 c_{1,t} \\ \kappa_2 c_{2,t} \end{pmatrix} + \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (49)$$

And introducing the following notations

$$V^{-1}\begin{pmatrix} \kappa_1 c_{1,t} \\ \kappa_2 c_{2,t} \end{pmatrix} = \begin{pmatrix} \epsilon_1(t) \\ \epsilon_2(t) \end{pmatrix},$$

leads to the solutions for the system (49):

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \int_s^t \epsilon_1(u)e^{\gamma_1(t-u)}du \\ \int_s^t \epsilon_2(u)e^{\gamma_2(t-u)}du \end{pmatrix} + \begin{pmatrix} e^{\gamma_1(t-s)} & 0 \\ 0 & e^{\gamma_2(t-s)} \end{pmatrix} V^{-1}\begin{pmatrix} \lambda_s^1 \\ \lambda_s^2 \end{pmatrix}$$

that allows us to infer the expression (15) for moments of λ_t^i. Notice that the determinant Υ is always real and if parameters of mutual excitations $\delta_{1,2}$, $\delta_{2,1}$, are positive. As $\mu_1, \mu_2 > 0$, the determinant is also strictly positive and the matric V is invertible. Finally, equation (15) states that conditonally to the sample path of the Markov chain θ_t, processes λ_t^1 and λ_t^2 are Markov given that their $\mathcal{F}_s \vee \mathcal{G}_t-$expectations only depend on the pair $\left(\lambda_t^1, \lambda_t^2\right)$.■

Proof of proposition 3.2 From the previous proposition, we infer that the unconditional expectations of OAI are the solutions of the following system

$$\begin{aligned}\begin{pmatrix} \mathbb{E}\left(\lambda_t^1 \,|\, \mathcal{F}_s\right) \\ \mathbb{E}\left(\lambda_t^2 \,|\, \mathcal{F}_s\right) \end{pmatrix} &= V \int_s^t \begin{pmatrix} e^{\gamma_1(t-u)} & 0 \\ 0 & e^{\gamma_2(t-u)} \end{pmatrix} V^{-1} \begin{pmatrix} \kappa_1 \mathbb{E}\left(c_{1,u} \,|\, \mathcal{F}_s\right) \\ \kappa_2 \mathbb{E}\left(c_{2,u} \,|\, \mathcal{F}_s\right) \end{pmatrix} du \\ &+ V \begin{pmatrix} e^{\gamma_1(t-s)} & 0 \\ 0 & e^{\gamma_2(t-s)} \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_s^1 \\ \lambda_s^2 \end{pmatrix}.\end{aligned} \tag{50}$$

Given that θ_t is a finite state Markov chain of generator Q_0 and if we remember that $c_i = \begin{pmatrix} c_{i,1} \\ \vdots \\ c_{i,l} \end{pmatrix}$ for $i = 1, 2$ are l- vectors, the expected level of mean reversion at time u is equal to:

$$\mathbb{E}\left(c_{i,u} | \mathcal{F}_s\right) = \theta_s^\top \exp\left(Q_0\left(u-s\right)\right) c_i$$

then expectation of intensities, conditionally to $\mathcal{F}_s$:

$$\begin{aligned}\begin{pmatrix} \mathbb{E}\left(\lambda_t^1 \,|\, \mathcal{F}_s\right) \\ \mathbb{E}\left(\lambda_t^2 \,|\, \mathcal{F}_s\right) \end{pmatrix} &= V \int_s^t \begin{pmatrix} e^{\gamma_1(t-u)} & 0 \\ 0 & e^{\gamma_2(t-u)} \end{pmatrix} V^{-1} \begin{pmatrix} \kappa_1 \theta_s^\top \exp\left(Q_0(u-s)\right) c_1 \\ \kappa_2 \theta_s^\top \exp\left(Q_0(u-s)\right) c_2 \end{pmatrix} du \\ &+ V \begin{pmatrix} e^{\gamma_1(t-s)} & 0 \\ 0 & e^{\gamma_2(t-s)} \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_s^1 \\ \lambda_s^2 \end{pmatrix}.\end{aligned} \tag{51}$$

If we replace V^{-1} by its definition (17), we obtain that

$$\begin{aligned} & V^{-1} \begin{pmatrix} \kappa_1 \theta_s^\top \exp\left(Q_0\left(u-s\right)\right) c_1 \\ \kappa_2 \theta_s^\top \exp\left(Q_0\left(u-s\right)\right) c_2 \end{pmatrix} \\ &= \frac{1}{\Upsilon} \begin{pmatrix} \begin{pmatrix} \kappa_1\left(\left(\delta_{1,1}\mu_1 - \kappa_1\right) - \gamma_2\right) \theta_s^\top \exp\left(Q_0\left(u-s\right)\right) c_1 \\ +\kappa_2 \delta_{1,2}\mu_2 \theta_s^\top \exp\left(Q_0\left(u-s\right)\right) c_2 \end{pmatrix} \\ \begin{pmatrix} \kappa_1\left(\gamma_1 - \left(\delta_{1,1}\mu_1 - \kappa_1\right)\right) \theta_s^\top \exp\left(Q_0\left(u-s\right)\right) c_1 \\ -\kappa_2 \delta_{1,2}\mu_2 \theta_s^\top \exp\left(Q_0\left(u-s\right)\right) c_2 \end{pmatrix} \end{pmatrix}. \end{aligned}$$

The integrand in equation (51) becomes then:

$$\begin{aligned} & \begin{pmatrix} e^{\gamma_1(t-u)} & 0 \\ 0 & e^{\gamma_2(t-u)} \end{pmatrix} V^{-1} \begin{pmatrix} \kappa_1 \theta_s^\top \exp\left(Q_0(u-s)\right) c_1 \\ \kappa_2 \theta_s^\top \exp\left(Q_0(u-s)\right) c_2 \end{pmatrix} \\ &= \frac{1}{\Upsilon} \begin{pmatrix} \begin{pmatrix} e^{\gamma_1 t} \kappa_1\left(\left(\delta_{1,1}\mu_1 - \kappa_1\right) - \gamma_2\right) \theta_s^\top \exp\left(\left(Q_0 - \gamma_1 I\right)\left(u-s\right)\right) c_1 \\ +e^{\gamma_1 t} \kappa_2 \delta_{1,2}\mu_2 \theta_s^\top \exp\left(\left(Q_0 - \gamma_1 I\right)\left(u-s\right)\right) c_2 \end{pmatrix} \\ \begin{pmatrix} e^{\gamma_2 t} \kappa_1\left(\gamma_1 - \left(\delta_{1,1}\mu_1 - \kappa_1\right)\right) \theta_s^\top \exp\left(\left(Q_0 - \gamma_2 I\right)\left(u-s\right)\right) c_1 \\ -e^{\gamma_2 t} \kappa_2 \delta_{1,2}\mu_2 \theta_s^\top \exp\left(\left(Q_0 - \gamma_2 I\right)\left(u-s\right)\right) c_2 \end{pmatrix} \end{pmatrix}. \end{aligned}$$

and we can conclude by direct integration that expected value of λ_t^i are given by equation (19). This result also states processes λ_t^1 and λ_t^2 are Markov given that their $\mathcal{F}_s$expectations only depend on the information available at time s: $\left(\lambda_s^1, \lambda_s^2, \theta_s^1, \theta_s^2\right)$. ■

Proof of Corollary 3.3
To prove this statement, it is sufficient to show that the conditional expectation of these processes with respect to $\mathcal{F}_s$ depends exclusively upon the information available at time s. Using the Tower property of conditional expectation, the expected number of supply order conditionally to $\mathcal{F}_s$ is then equal to the following product:

$$\mathbb{E}\left(N_t^1|\mathcal{F}_s\right) = \mathbb{E}\left(\mathbb{E}\left(N_t^1|\mathcal{F}_s \vee \mathcal{H}_t\right)|\mathcal{F}_s\right).$$

By construction, the compensator of process N_t^1 is an $\mathcal{H}_t$-adapted process $\int_0^t \lambda_u^1 du$ such that the compensated process $M_t^1 = N_t^1 - \int_0^t \lambda_u^1 du$ is a martingale. Given that $\mathbb{E}\left(M_t^1|\mathcal{F}_s \vee \mathcal{H}_t\right) = M_s^1$, we deduce that $\mathbb{E}\left(N_t^1|\mathcal{F}_s \vee \mathcal{H}_t\right) = N_s^1 + \int_s^t \lambda_u^1 du$. Using the Fubini's theorem, we infer that

$$\begin{aligned}\mathbb{E}\left(N_t^1|\mathcal{F}_s\right) &= \left(N_s^1 + \mathbb{E}\left(\int_s^t \lambda_u^1 du|\mathcal{F}_s\right)\right) \qquad (52)\\ &= N_s^1 + \int_s^t \mathbb{E}\left(\lambda_u^1|\mathcal{F}_s\right) du\,.\end{aligned}$$

According to proposition 3.2, $\mathbb{E}\left(\lambda_u^1|\mathcal{F}_s\right)$ depends only upon λ_s^1, λ_s^2 and θ_s. From equation (52), we immediately deduce that $\mathbb{E}\left(N_t^1|\mathcal{F}_s\right)$ is exclusively a function of $(\lambda_s^1,\, \lambda_s^2\, ,\theta_s,\, N_s^1)$. The same holds for N_t^2. By definition, L_t^1 is a sum of independent random variables:

$$\begin{aligned}\mathbb{E}\left(L_t^1|\mathcal{F}_s\right) &= \mathbb{E}\left(\sum_{n=1}^{N_t^1} O_n^1|\mathcal{F}_s\right)\\ &= \mu_1 \mathbb{E}\left(N_t^1|\mathcal{F}_s\right).\end{aligned}$$

As $\mathbb{E}\left(N_t^1|\mathcal{F}_s\right)$ is a function of $(\lambda_s^1,\, \lambda_s^2\, ,\theta_s,\, N_s^1)$, the same conclusion holds for $\mathbb{E}\left(L_t^1|\mathcal{F}_s\right)$. A similar reasoning for L_t^2 and proposition 3.2 allows to end the proof. ■

Proof of proposition 3.4 If we remember the equation (48), we infer that the expectations of $c_{j,t}\lambda_t^i$ for $i,j=1,2$ are solution of ordinary differential equations (ODE):

$$\underbrace{\begin{pmatrix} \frac{\partial}{\partial t}\mathbb{E}\left(c_{1,t}\lambda_t^1|\mathcal{F}_0 \vee \mathcal{G}_t\right) \\ \frac{\partial}{\partial t}\mathbb{E}\left(c_{2,t}\lambda_t^1|\mathcal{F}_0 \vee \mathcal{G}_t\right) \\ \frac{\partial}{\partial t}\mathbb{E}\left(c_{1,t}\lambda_t^2|\mathcal{F}_0 \vee \mathcal{G}_t\right) \\ \frac{\partial}{\partial t}\mathbb{E}\left(c_{2,t}\lambda_t^2|\mathcal{F}_0 \vee \mathcal{G}_t\right) \end{pmatrix}}_{:=dE(t)} = \underbrace{\begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_1 \\ 0 & 0 & \kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix}}_{:=K} \underbrace{\begin{pmatrix} c_{1,t}^2 \\ c_{2,t}^2 \\ c_{1,t}c_{2,t} \end{pmatrix}}_{:=C_t^2} +$$

$$\underbrace{\begin{pmatrix} \delta_{1,1}\mu_1 - \kappa_1 & 0 & \delta_{1,2}\mu_2 & 0 \\ 0 & \delta_{1,1}\mu_1 - \kappa_1 & 0 & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & 0 & \delta_{2,2}\mu_2 - \kappa_2 & 0 \\ 0 & \delta_{2,1}\mu_1 & 0 & \delta_{2,2}\mu_2 - \kappa_2 \end{pmatrix}}_{WFW^{-1}} \underbrace{\begin{pmatrix} \mathbb{E}\left(c_{1,t}\lambda_t^1|\mathcal{F}_0 \vee \mathcal{G}_t\right) \\ \mathbb{E}\left(c_{2,t}\lambda_t^1|\mathcal{F}_0 \vee \mathcal{G}_t\right) \\ \mathbb{E}\left(c_{1,t}\lambda_t^2|\mathcal{F}_0 \vee \mathcal{G}_t\right) \\ \mathbb{E}\left(c_{2,t}\lambda_t^2|\mathcal{F}_0 \vee \mathcal{G}_t\right) \end{pmatrix}}_{E(t)}.$$

We summarize this system of ODE as follows

$$dE(t) = K\, C_t^2 + W\, F\, W^{-1} E(t)\,.$$

If we note $U(t) = W^{-1}E(t)$, we rewrite this last system:

$$dU(t) \quad = \quad W^{-1}K\, C_t^2 + F\, U(t)\,,$$

that admits the following solution:

$$U(t) \quad = \quad \int_0^t \exp\left(F\, s\right)\, W^{-1}K\, C_s^2\, ds + \exp\left(F\, t\right) U(0)\,,$$

and we can conclude.■

Proof of proposition 3.5. If we remember equation (27), we can develop it as follows

$$\begin{aligned}
&\exp\left(F\, s\right) W^{-1} \begin{pmatrix} \kappa_1 c_{1,s}^2 \\ \kappa_1 c_{1,s} c_{2,s} \\ \kappa_2 c_{1,s} c_{2,s} \\ \kappa_2 c_{2,s}^2 \end{pmatrix} \\
&= \frac{1}{\Upsilon} \begin{pmatrix} \kappa_1\left(\left(\delta_{1,1}\mu_1 - \kappa_1\right) - \gamma_2\right)\left(e^{\gamma_1 s} c_{1,s}^2\right) + \delta_{1,2}\mu_2\kappa_2\left(e^{\gamma_1 s} c_{1,s} c_{2,s}\right) \\ \kappa_1\left(\gamma_1 - \left(\delta_{1,1}\mu_1 - \kappa_1\right)\right)\left(e^{\gamma_2 s} c_{1,s}^2\right) - \delta_{1,2}\mu_2\kappa_2\left(e^{\gamma_2 s} c_{1,s} c_{2,s}\right) \\ \kappa_1\left(\left(\delta_{1,1}\mu_1 - \kappa_1\right) - \gamma_2\right)\left(e^{\gamma_1 s} c_{1,s} c_{2,s}\right) + \delta_{1,2}\mu_2\kappa_2\left(e^{\gamma_1 s} c_{2,s}^2\right) \\ \kappa_1\left(\gamma_1 - \left(\delta_{1,1}\mu_1 - \kappa_1\right)\right)\left(e^{\gamma_2 s} c_{1,s} c_{2,s}\right) - \delta_{1,2}\mu_2\kappa_2\left(e^{\gamma_2 s} c_{2,s}^2\right) \end{pmatrix},
\end{aligned}$$

and its expectation is given by

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(F\, s\right) W^{-1} \begin{pmatrix} \kappa_1 c_{1,s}^2 \\ \kappa_1 c_{1,s} c_{2,s} \\ \kappa_2 c_{1,s} c_{2,s} \\ \kappa_2 c_{2,s}^2 \end{pmatrix}\right) \\
&= \frac{1}{\Upsilon} \begin{pmatrix} \kappa_1\left(\left(\delta_{1,1}\mu_1 - \kappa_1\right) - \gamma_2\right)\left(\theta_0 e^{(Q_0+\gamma_1 I)s}\bar{c}_1^2\right) + \delta_{1,2}\mu_2\kappa_2\left(\theta_0 e^{(Q_0+\gamma_1 I)s}\bar{c}_{1,2}\right) \\ \kappa_1\left(\gamma_1 - \left(\delta_{1,1}\mu_1 - \kappa_1\right)\right)\left(\theta_0 e^{(Q_0+\gamma_2 I)s}\bar{c}_1^2\right) - \delta_{1,2}\mu_2\kappa_2\left(\theta_0 e^{(Q_0+\gamma_2 I)s}\bar{c}_{1,2}\right) \\ \kappa_1\left(\left(\delta_{1,1}\mu_1 - \kappa_1\right) - \gamma_2\right)\left(\theta_0 e^{(Q_0+\gamma_1 I)s}\bar{c}_{1,2}\right) + \delta_{1,2}\mu_2\kappa_2\left(\theta_0 e^{(Q_0+\gamma_1 I)s}\bar{c}_2^2\right) \\ \kappa_1\left(\gamma_1 - \left(\delta_{1,1}\mu_1 - \kappa_1\right)\right)\left(\theta_0 e^{(Q_0+\gamma_2 I)s}\bar{c}_{1,2}\right) - \delta_{1,2}\mu_2\kappa_2\left(\theta_0 e^{(Q_0+\gamma_2 I)s}\bar{c}_2^2\right) \end{pmatrix}.
\end{aligned}$$

Integrating this last equation allows us to conclude. ■

Proof of proposition 3.6. If we remember the expression (24) of the infinitesimal generator, we have

$$\begin{aligned}
\mathcal{A}\left(\left(\lambda_t^1\right)^2\right) &= 2\kappa_1(c_{1,t}-\lambda_t^1)\lambda_t^1+\lambda_t^1\int_{-\infty}^{+\infty}\left(\lambda_t^1+\delta_{1,1}z\right)^2-\left(\lambda_t^1\right)^2\nu_1(dz)\\
&\quad+\lambda_t^2\int_{-\infty}^{+\infty}\left(\lambda_t^1+\delta_{1,2}z\right)^2-\left(\lambda_t^1\right)^2\nu_2(dz)\,,\\
\mathcal{A}\left(\left(\lambda_t^2\right)^2\right) &= 2\kappa_2(c_{2,t}-\lambda_t^2)\lambda_t^2+\lambda_t^1\int_{-\infty}^{+\infty}\left(\lambda_t^2+\delta_{2,1}z\right)^2-\left(\lambda_t^2\right)^2\nu_1(dz)\\
&\quad+\lambda_t^2\int_{-\infty}^{+\infty}\left(\lambda_t^2+\delta_{2,2}z\right)^2-\left(\lambda_t^2\right)^2\nu_2(dz)\,,\\
\mathcal{A}\left(\lambda_t^1\lambda_t^2\right) &= \kappa_1(c_{1,t}-\lambda_t^1)\lambda_t^2+\kappa_2(c_{2,t}-\lambda_t^2)\lambda_t^1\\
&\quad+\lambda_t^1\int_{-\infty}^{+\infty}\left(\lambda_t^1+\delta_{1,1}z\right)\left(\lambda_t^2+\delta_{2,1}z\right)-\lambda_t^1\lambda_t^2\nu_1(dz)\\
&\quad+\lambda_t^2\int_{-\infty}^{+\infty}\left(\lambda_t^1+\delta_{1,2}z\right)\left(\lambda_t^2+\delta_{2,2}z\right)-\lambda_t^1\lambda_t^2\nu_2(dz)\,.
\end{aligned}$$

And given that $\frac{\partial}{\partial t}g=\mathbb{E}\left(\mathcal{A}g\,|\,\mathcal{F}_0\right)$, we can conclude.■

Proof of proposition 3.9. Let us assume that $\theta_t=e_i$. If we denote by $g(\lambda_t^1,J_t^1,\lambda_t^2,J_t^2,\theta_t)=\mathbb{E}\left(\omega^{N_T^k}\,|\,\mathcal{F}_t\right)$, g is solution of the following Itô's equation for semi martingale :

$$\begin{aligned}
0 &= g_t+\kappa_1(c_{1,t}-\lambda_t^1)g_{\lambda^1}+\kappa_2(c_{2,t}-\lambda_t^2)g_{\lambda^2} \qquad (53)\\
&\quad+\lambda_t^1\int_{-\infty}^{+\infty}g(\lambda_t^1+\delta_{1,1}z,J_t^1+(z,1)^\top,\lambda_t^2+\delta_{2,1}z,J_t^2,e_i)-g(\lambda_t^1,J_t^1,\lambda_t^2,J_t^2,e_i)d\nu_1(z)\\
&\quad+\lambda_t^2\int_{-\infty}^{+\infty}g(\lambda_t^1+\delta_{1,2}z,J_t^1,\lambda_t^2+\delta_{2,2}z,J_t^2+(z,1)^\top,e_i)-g(\lambda_t^1,J_t^1,\lambda_t^2,J_t^2,e_i)d\nu_2(z)\\
&\quad+\sum_{j\neq i}^{N}q_{i,j}\left(g(\lambda_t^1,J_t^1,\lambda_t^2,J_t^2,e_j)-g(\lambda_t^1,J_t^1,\lambda_t^2,J_t^2,e_i)\right)\,.
\end{aligned}$$

Next, we assume that g is an exponential affine function of λ_t^1, λ_t^2 and N_t^i :

$$g = \exp\left(A(t,T,e_j)+B_1(t,T)\lambda_t^1+B_2(t,T)\lambda_t^2+C(t,T)N_t^k\right),$$

where $A(t,T,e_i)$ for $i=1$ to l, $B_1(t,T)$, $B_2(t,T)$ and $C(t,T)$ are time dependent functions. The partial derivatives of g are then given by:

$$g_t = \left(\frac{\partial}{\partial t}A(t,T,e_j)+\frac{\partial}{\partial t}B_1(t,T)\lambda_t^1+\frac{\partial}{\partial t}B_2(t,T)\lambda_t^2+\frac{\partial}{\partial t}C(t,T)N_t^k\right)g,$$

$$g_{\lambda^1}=B_1(t,T)g \qquad \textit{and} \quad g_{\lambda^2}=B_2(t,T)g\,.$$

And the integrands in equation (53) are rewritten with the notations $A:=A(t,T,e_i)$, $B_1:=B_1(t,T)$, $B_2:=B_2(t,T)$ and $C:=C(t,T)$ as follows:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}g(\lambda_t^1+\delta_{1,1}z,J_t^1+(z,1)^\top,\lambda_t^2+\delta_{2,1}z,J_t^2,e_i)-g(\lambda_t^1,J_t^1,\lambda_t^2,J_t^2,e_i)\,d\nu_1(z)\\
&\quad= g\left[e^{1_{k=1}C}\psi_1\left(B_1\delta_{1,1}+B_2\delta_{2,1}\right)-1\right],
\end{aligned}$$

$$\int_{-\infty}^{+\infty} g(\lambda_t^1 + \delta_{1,2} z, J_t^1, \lambda_t^2 + \delta_{2,2} z, J_t^2 + (z,1)^\top, e_j) - g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_i)\, d\nu_2(z)$$
$$= g\left[e^{1_{k=2} C} \psi_2\left(B_1 \delta_{1,2} + B_2 \delta_{2,2}\right) - 1\right].$$

As the sum of instantaneous probabilities is null, $q_{ii} = -\sum_{i\neq j}^{l} q_{i,j}$, we have that

$$\sum_{j\neq i}^{l} q_{i,j}\left(g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_j) - g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_i)\right) = \sum_{j=1}^{l} q_{i,j} g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_j).$$

Then the equation (53) becomes:

$$\begin{aligned} 0 = &\left(\frac{\partial}{\partial t} A + \frac{\partial}{\partial t} B_1 \lambda_t^1 + \frac{\partial}{\partial t} B_2 \lambda_t^2 + \frac{\partial}{\partial t} C\, N_t^i\right) e^{A(t,T,e_i)} \\ &+\kappa_1 (c_{1,t} - \lambda_t^1) B_1\, e^{A(t,T,e_i)} + \kappa_2 (c_{2,t} - \lambda_t^2) B_2\, e^{A(t,T,e_i)} \\ &+\lambda_t^1 \left(e^{1_{k=1} C} \psi_1\left(B_1 \delta_{1,1} + B_2 \delta_{2,1}\right) - 1\right) e^{A(t,T,e_i)} \\ &+\lambda_t^2 \left(e^{1_{k=2} C} \psi_2\left(B_1 \delta_{1,2} + B_2 \delta_{2,2}\right) - 1\right) e^{A(t,T,e_i)} \\ &+\sum_{j=1}^{l} q_{i,j} g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2, e_j), \end{aligned} \tag{54}$$

from which we guess that $C(t,s) = \ln \omega$. Regrouping terms allows to infer that

$$\begin{aligned} 0 = &\frac{\partial}{\partial t} A e^{A(t,T,e_i)} + \kappa_1 c_{1,t}\, B_1\, e^{A(t,T,e_i)} + \kappa_2 c_{2,t}\, B_2\, e^{A(t,T,e_i)} + \sum_{j=1}^{l} q_{i,j} e^{A(t,T,e_j)} \\ &+\lambda_t^1 \left(\frac{\partial}{\partial t} B_1 - \kappa_1 B_1 + \left[1_{k=1} \omega \psi_1\left(B_1 \delta_{1,1} + B_2 \delta_{2,1}\right) - 1\right]\right) e^{A(t,T,e_i)} \\ &+\lambda_t^2 \left(\frac{\partial}{\partial t} B_2 - \kappa_2 B_2 + \left[1_{k=2} \omega \psi_2\left(B_1 \delta_{1,2} + B_2 \delta_{2,2}\right) - 1\right]\right) e^{A(t,T,e_i)}. \end{aligned}$$

Given that λ_t^1 and λ_t^2 are random quantities, this equation is satisfied if and only if

$$\begin{aligned} \frac{\partial}{\partial t} B_1 &= \kappa_1 B_1 - \left[1_{k=1} \omega \psi_1\left(B_1 \delta_{1,1} + B_2 \delta_{2,1}\right) - 1\right] \\ \frac{\partial}{\partial t} B_2 &= \kappa_2 B_2 - \left[1_{k=2} \omega \psi_2\left(B_1 \delta_{1,2} + B_2 \delta_{2,2}\right) - 1\right] \\ \left(\frac{\partial}{\partial t} A\right) e^{A(t,T,e_i)} &= -\kappa_1 c_{1,t}\, B_1\, e^{A(t,T,e_i)} - \kappa_2 c_{2,t}\, B_2\, e^{A(t,T,e_i)} - \sum_{j=1}^{l} q_{i,j} e^{A(t,T,e_j)}. \end{aligned}$$

If we define $\tilde{A}(t,T) = \left(e^{A(t,T,e_1)}, ..., e^{A(t,T,e_l)}\right)$, the last equations can finally be put in matrix form as:

$$\frac{\partial \tilde{A}}{\partial t} + \left(\operatorname{diag}\left(\kappa_1 c_{1,t}\, B_1 + \kappa_2 c_{2,t}\, B_2\right) + Q_0\right) \tilde{A} = 0.$$

■

Proof of proposition 3.11. From previous results, we know that $B_k(t,T)$ is solution of the following ODE

$$\frac{\partial}{\partial t} B_k = \kappa_k B_k + (-1)^k \omega_1 \alpha_k \mu_k - \left[\psi_k\left(B_1 \delta_{1,k} + B_2 \delta_{2,k} + C_k\right) - 1\right], \quad k = 1, 2$$

with terminal condition $B_k(T,T)=\omega_{k+1}$. If we set $B_k(t,T)=D_k(T-t)$ and $\tau=T-t$. Then

$$\frac{\partial B_k}{\partial t}=\frac{\partial D_k}{\partial \tau}\frac{\partial \tau}{\partial t}=-\frac{\partial D_k}{\partial \tau}\,.$$

Thus we obtain

$$\begin{aligned}\frac{\partial D_k}{\partial \tau} &= -\kappa_k B_k(\tau)-(-1)^k\,\omega_1\alpha_k\mu_k+\left[\psi_k\left(D_1(\tau)\delta_{1,k}+D_2(\tau)\delta_{2,k}+C_k\right)-1\right] \\ &= -\kappa_k D_k(\tau)+\psi_k\left(D_1(\tau)\delta_{1,k}+D_2(\tau)\delta_{2,k}+C_k\right)-\left[(-1)^k\,\omega_1\alpha_k\mu_k+1\right] \\ &= -\kappa_k D_k(\tau)+\psi_k\left(D_1(\tau)\delta_{1,k}+D_2(\tau)\delta_{2,k}+C_k\right)-\beta_k(\omega_1)\,.\end{aligned} \tag{55}$$

The left hand side is then denoted $h_k(D_1,D_2)$. Due to the convexity of ψ_k there is only one point (u_1^*,u_2^*) such that $h_k(u)=0$ for $k=1,2$. These equations are indeed equivalent to

$$\psi_k\left(u_1\delta_{1,k}+u_2\delta_{2,k}+C_k\right)=\beta_k(\omega_1)+\kappa_k u_k\,.$$

We rewrite the equations (55) as follows,

$$\frac{dD_k}{-\kappa_k D_k+\psi_k\left(D_1\delta_{1,k}+D_2\delta_{2,k}+C_k\right)-\beta_k(\omega_1)}=d\tau\,.$$

As $D_k(0)=\omega_{k+1}$ for $k\in\{1,2\}$ by direct integration, we have that

$$\int_{\omega_2}^{D_1}\frac{du_1}{-\kappa_1 u_1+\psi_1\left(u_1\delta_{1,1}+D_2\delta_{2,1}+C_1\right)-\beta_1(\omega_1)}=\tau\,,$$

$$\int_{\omega_3}^{D_2}\frac{du_2}{-\kappa_2 u_2+\psi_2\left(D_1\delta_{1,2}+u_2\delta_{2,2}+C_2\right)-\beta_2(\omega_1)}=\tau\,.$$

with $D_k\in[\omega k+1,u_k^*)$ or $D_k\in[u_k^*,\omega k+1)$.

We can remark that if $(D_1,D_2)=(u_1^*,u_2^*)$ then $\tau=+\infty$ as the numerator converges to zero. If we define the functions $F^1_{\omega_1}(x,y)$ and $F^2_{\omega_1}(x,y)$ from $\mathbb{R}^2$ to $\mathbb{R}^+$ by equations (36), D_1 and D_2 are such that $F^k_{\omega_1}(D_1,D_2)=\tau$. If $\left(F^1_{\omega_1}\right)^{-1}(\tau\,|\,y)$ and $\left(F^2_{\omega_1}\right)^{-1}(\tau\,|\,x)$ are respectively the inverse functions of $F^1_{\omega_1}(.,y)$ and $F^2_{\omega_1}(x,.)$, then D_1 and D_2 satisfy the following system

$$\begin{aligned}D_1 &= \left(F^1_{\omega_1}\right)^{-1}(\tau\,|\,D_2)\,,\\ D_2 &= \left(F^2_{\omega_1}\right)^{-1}(\tau\,|\,D_1)\,,\end{aligned}$$

or $B_1(t,T)=\left(F^1_{\omega_1}\right)^{-1}(T-t\,|\,B_2(t,T))$ and $B_2(t,T)=\left(F^2_{\omega_1}\right)^{-1}(T-t\,|\,B_1(t,T))$. ■

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